St. Venant Torsion

Torque in structural members is carried by shear stress, i.e., St. Venant torsion, and possibly axial stress, i.e., warping torsion. This document addresses the former, employing a “stress function” to obtain the solution. From this perspective, the naming of the theory after Adhémar Jean Claude Barré de Saint-Venant (1797-1886) is perhaps somewhat misleading. This is because St. Venant formulated the problem in terms of an unknown function that represents the axial displacement at any point of the cross-section. It was Ludwig Prandtl (1875-1953) who reformulated the problem in terms of an unknown stress function, which is the one addressed in this document. Two primary objectives are identified in this document; to compute the cross-sectional constant for St. Venant torsion and to compute stresses in the cross-section for a given torque.

Axisymmetric Cross-section

Consider first the rather simple case of an axisymmetric cross-section, e.g., a circular cross-section.

Equilibrium

By considering an infinitesimally short element with length dx, subjected to a distributed torque with intensity m_x, one obtains:

$$m_x = -\frac{dT}{dx} \tag{1}$$

Section Integration

Integration of shear stresses multiplied by their distance to the centre yields:

$$T = \int_{r_i}^{r_o} (2\pi r) \cdot \tau \cdot r \cdot \mathrm{d}r \tag{2}$$

where r_i and r_o are the inner and outer radii of the cross-section.

Material Law

Hooke’s law for shear stresses and strains reads:

$$\tau = G \cdot \gamma \tag{3}$$

where G is the shear modulus, $G=E/(2(1+\nu))$.

Kinematics

The relationship between shear strain, γ, and the rotation of the cross-section, ϕ, is obtained by expressing the length of the line segment identified by an arrow in Figure 1 in two ways: $dx \cdot \gamma = d\phi \cdot r$, which leads to the kinematics equation

$$\frac{\mathrm{d}\phi}{\mathrm{d}x} = \frac{\gamma}{r} \tag{4}$$

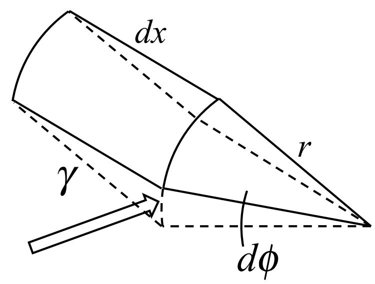

Figure 1: Kinematics for an axisymmetric cross-section subjected to torsion.

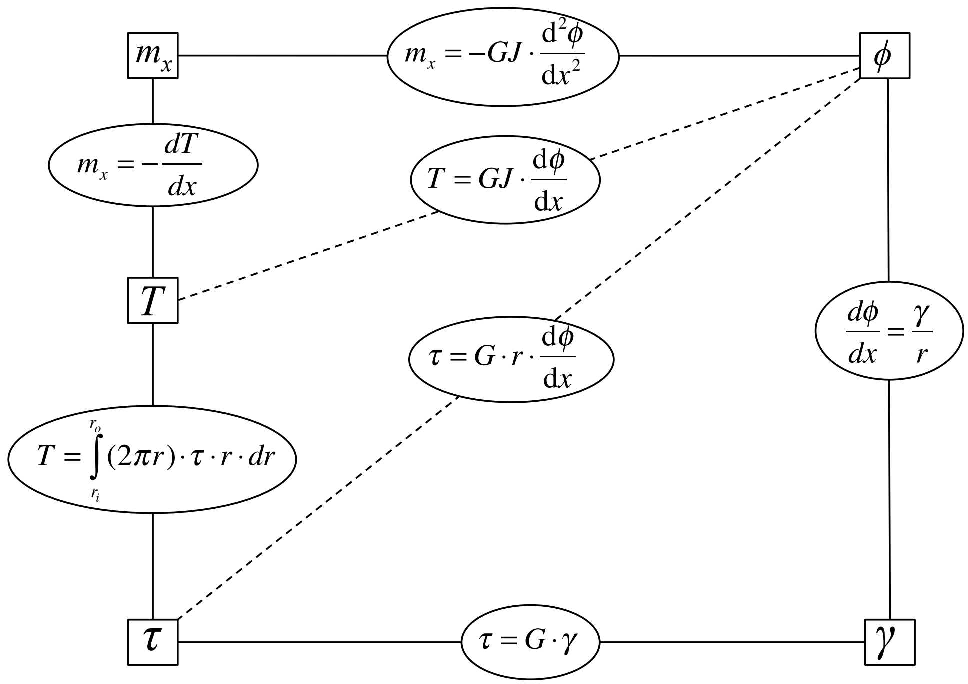

Figure 2: Governing equations for axisymmetric cross-sections.

Differential Equation

The differential equation is obtained by combining all the previous equations, which are summarized in Figure 2:

$$m_x = -G \cdot J \cdot \frac{d^2\phi}{dx^2} \tag{5}$$

where the following definition has been made:

$$J = \int_{r_i}^{r_o} 2\pi r^3 \, dr = \frac{\pi}{2} \cdot \left(r_o^4 - r_i^4 \right) \tag{6}$$

J is the cross-sectional constant for St. Venant torsion, and is sometimes denoted I_p in other literature. If the equilibrium equations are omitted then the differential equation reads:

$$T = GJ \cdot \frac{d\phi}{dx} \tag{7}$$

General Solution

Integrating the differential equation twice yields the general solution:

$$\phi = -\frac{m_x}{2 \cdot GJ} \cdot x^2 + C_1 \cdot x + C_2 \tag{8}$$

Shear Stress

To obtain an expression for the shear stress in terms of the stress resultant, the material law and kinematics equations are first combined:

$$\tau = G \cdot r \cdot \frac{d\phi}{dx} \tag{9}$$

Then the following differential equation that encompasses all of material law, kinematics, and stress resultant is considered:

$$T = GJ \cdot \frac{d\phi}{dx} \tag{10}$$

Substitution of Eq. (10) into Eq. (9) yields the sought stress:

$$\tau = \frac{T}{J} \cdot r \tag{11}$$

The expression shows that the shear stress increases outwards, proportional to the radius, i.e., the distance from the centre to the considered fibre.

Arbitrary Cross-sections

General equations for St. Venant torsion are established here, valid for both massive and thin-walled cross-sections. Subsequent pages specialize the equations to particular cross-section types.

Equilibrium and Prandtl's Stress Function

The equilibrium between externally applied distributed torque, m_x, and stress resultant, T, expressed in Eq. (1) remains valid in all St. Venant torsion. However, it is the internal equilibrium of stresses at a point that requires special attention. According to mechanics of solids, angular momentum equations yield the equality of shear stress pairs. Conversely, equilibrium of linear momentum at any point in the material impose the following condition:

$$\sigma_{ij,i} = 0 \quad \Rightarrow \quad \begin{aligned} \sigma_{xx,x} + \tau_{yx,y} + \tau_{zx,z} &= 0 \\ \tau_{xy,x} + \sigma_{yy,y} + \tau_{zy,z} &= 0 \\ \tau_{xz,x} + \tau_{yz,y} + \sigma_{zz,z} &= 0 \end{aligned} \tag{12}$$

where internal body forces are neglected. Instead of working directly with these stress components, Prandtl introduced a new function that plays a central role in the theory of torsion. The function is called Prandtl's stress function, denoted $P(y,z)$. By itself, the function has no physical meaning, but its all-important feature is that stresses are derived from it. Specifically, Prandtl's stress function is *defined* such that

$$\begin{aligned}\tau_{xy} &= \frac{\partial P}{\partial z} = P_{,z} \\ \tau_{xz} &= -\frac{\partial P}{\partial y} = -P_{,y}\end{aligned} \tag{13}$$

It is observed that the shear stress in one direction is obtained by differentiating the stress function in the perpendicular direction. It is also noted that φ varies only with the cross-section coordinates y and z but it does not vary with x. Substitution of Eq. (13) into Eq. (12) yields:

$$\begin{aligned}\sigma_{xx,x} + \tau_{yx,y} + \tau_{zx,z} &= 0 + P_{,zy} - P_{,yz} = 0 \\ \tau_{xy,x} + \sigma_{yy,y} + \tau_{zy,z} &= P_{,zx} + 0 + 0 = 0 \\ \tau_{xz,x} + \tau_{yz,y} + \sigma_{zz,z} &= -P_{,zx} + 0 + 0 = 0\end{aligned} \tag{14}$$

because 1) the stress function does not vary with x; 2) all axial stresses are zero in St. Venant theory; and 3) the shear stress τ_{yz} is zero because the shear strain γ_{yz} is zero when the cross-section is assumed to retain its shape. As a result, the use of Prandtl's stress function as a measure of stress automatically satisfies the equilibrium equations.

Boundary Conditions for the Stress Function

The shear stress on the surface of a structural member is obviously zero. This translates into a boundary condition for the stress function. To formulate this boundary condition mathematically, let s be the coordinate that follows the edge of the cross-section, let r be the axis that is perpendicular to s, and let α denote the angle between the y-axis and the edge of the cross-section, as illustrated in Figure 3.

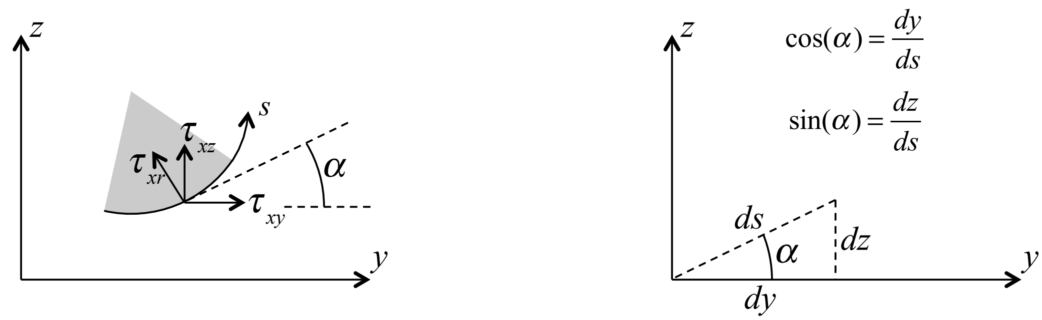

Figure 3: Shear stress perpendicular to the edge of the cross-section (left) and relationship between differentials (right).

Because there are no stresses on the free surface, it is required that, on the free surface:

$$\tau_{xr} = 0 \tag{15}$$

By the definition of the stress function, the shear stress in the r-direction is obtained by differentiating the stress function in the s-direction:

$$\tau_{xr} = \frac{\partial P(r,s)}{\partial s} \tag{16}$$

Thus, the fact that there are no stresses on the free surface translates into the following boundary condition for the stress function:

$$\frac{\partial P(r,s)}{\partial s} = 0 \tag{17}$$

Another way of deriving the same result is to use decomposition of the shear stresses τ_{xy} and τ_{xz}. Then, the condition of zero shear stress perpendicular to the edge of the cross-section reads:

$$\tau_{xr} = \tau_{xz} \cdot \cos(\alpha) - \tau_{xy} \cdot \sin(\alpha) = -\left(\frac{\partial P}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial P}{\partial z}\frac{\partial z}{\partial s} \right) = \frac{\partial P}{\partial s} = 0 \tag{18}$$

In short, the derivative of the stress function must be zero along the edge of the cross-section, which implies that the stress function must be constant. For convenience and without loss of generality, this constant is set equal to zero. When observing the simplicity of this boundary condition it is noted that St. Venant's formulation of the theory, in terms of a function that describes the axial displacement in the cross-section, introduces a more complicated equation for this boundary condition. With Prandtl's stress function one simply needs to assure a constant value along the edge of the cross-section.

Section Integration

Integration of shear stresses multiplied by their distance to the centre yields:

$$T = \int_A \left(\tau_{xz} \cdot y - \tau_{xy} \cdot z \right) \mathrm{d}A \tag{19}$$

When the stresses are formulated in terms of Prandtl's stress function then it takes the form:

$$T = -\int_A \left(\frac{\partial P}{\partial y} \cdot y + \frac{\partial P}{\partial z} \cdot z \right) \mathrm{d}A \tag{20}$$

This integral is simplified further by invoking integration by parts. The first term in the integrand is partially integrated in the y direction and the second term in z direction:

$$\begin{aligned} T &= -\left(\int_A \frac{\partial P}{\partial y} \cdot y \cdot \mathrm{d}A \right) - \left(\int_A \frac{\partial P}{\partial z} \cdot z \cdot \mathrm{d}A \right) \\ &= -\left(\oint P \cdot y \cdot \mathrm{d}\Gamma - \int_A P \cdot \mathrm{d}A \right) - \left(\oint P \cdot z \cdot \mathrm{d}\Gamma - \int_A P \cdot \mathrm{d}A \right) \\ &= -\underbrace{\oint P \cdot y \cdot \mathrm{d}\Gamma}_{0} - \underbrace{\oint P \cdot z \cdot \mathrm{d}\Gamma}_{0} + \int_A P \cdot \mathrm{d}A + \int_A P \cdot \mathrm{d}A \\ &= 2 \cdot \int_A P \cdot \mathrm{d}A \end{aligned} \tag{21}$$

where the boundary integrals vanish because the stress function is zero around the outer edges of the cross-section. Eq. (21) is a cornerstone of the torsion theory that is presented on the following pages, and it is important to understand that the factor "2" is generally valid. For certain cross-sections, such as the open thin-walled cross-sections, equilibrium of the shear flow appears to give a torque that is half the value of Eq. (21). To address this puzzle, it is first noted that Eq. (21) consists of two equal contributions from shear stress in two perpendicular directions. For certain simplified stress functions, such as the one used for thin-walled open cross-sections, the stress at the short ends is neglected, which causes the apparent anomaly. In actuality there are shear stresses at those locations; in fact, those contributions double the torque because their moment arm is large. In Chapter 10 of their book on theory of elasticity, Timoshenko and Goodier acknowledge that the small shear stresses that are sometimes neglected can have an appreciable effect because their moment arm is substantial. They also suggest further reading by mentioning that the question was cleared up by Lord Kelvin in Kelvin and Tait's Natural Philosophy, Vol. 2, page 267 (Timoshenko and Goodier 1969).

Material Law

Hooke's law for a 3D material point relates axial stresses to axial strains, and shear stresses to shear strains. However, not every material law equation is necessary for Saint Venant torsion. When considering kinematics, it will become apparent that all axial strains are zero. In turn, all axial stresses are zero, and hence the material law equations for axial strains/stresses are not needed. For shear strains/stresses, the material law reads:

$$\begin{aligned}\tau_{xy} &= G \cdot \gamma_{xy} \\ \tau_{xz} &= G \cdot \gamma_{xz} \\ \tau_{yz} &= G \cdot \gamma_{yz}\end{aligned} \tag{22}$$

Kinematics

As a fundamental kinematics postulation, it is assumed that the cross-section retains its shape during torsion:

$$\begin{aligned}v &= -\phi \cdot z \\ w &= \phi \cdot y\end{aligned} \tag{23}$$

It is also assumed that the axial displacement in the cross-section, u, only varies with y and z. In other words, u is independent of x. The original formulation by St. Venant goes to greater lengths to characterize the function $u(y,z)$. However, this is circumvented in the theory formulated by Prandtl in terms of the stress function introduced above. With these assumptions, the kinematics for general 3D problems, first for axial strains, reads:

$$\begin{aligned}\varepsilon_x &= \frac{du}{dx} = 0 \\ \varepsilon_y &= \frac{dv}{dy} = 0 \\ \varepsilon_z &= \frac{dw}{dz} = 0\end{aligned} \tag{24}$$

where the first equation equals zero because u is independent of x, the second and third equations equal zero because no deformation of the shape of the cross-section is allowed. For shear strains, the general kinematics equations are:

$$\begin{aligned}\gamma_{xy} &= \frac{dv}{dx}+\frac{du}{dy}=-\frac{d\phi}{dx}\cdot z+\frac{du}{dy}\\ \gamma_{xz} &= \frac{du}{dz}+\frac{dw}{dx}=\frac{du}{dz}+\frac{d\phi}{dx}\cdot y\\ \gamma_{yz} &= \frac{dw}{dy}+\frac{dv}{dz}=0\end{aligned} \tag{25}$$

where the last equation is zero because the cross-section does not change shape.

Differential Equation

Kinematics and material law equations combined with Prandtl's stress function in Eq. (13) yields:

$$\begin{aligned}P_{,z} &= \tau_{xy}=G\cdot\gamma_{xy}=G\cdot\left(-\phi_{,x}\cdot z+u_{,y}\right)\\ P_{,y} &= -\tau_{xz}=-G\cdot\gamma_{xz}=-G\cdot\left(u_{,z}+\phi_{,x}\cdot y\right)\end{aligned} \tag{26}$$

These two equations can be combined into one. By differentiating the first with respect to z and the second with respect to y the quantity $u_{,yz}$ becomes a common quantity that facilitates the merger, which yields:

$$\frac{\partial^2 P(y,z)}{\partial y^2}+\frac{\partial^2 P(y,z)}{\partial z^2}\equiv P_{,yy}+P_{,zz}\equiv\nabla^2 P(y,z)=-2\cdot G\cdot\phi' \tag{27}$$

This is the general differential equation for Saint Venant torsion of members with arbitrary cross-sections, when Prandtl's stress function is employed. Put another way, this is the differential equation that governs the stress function. However, it is noted that it does not include the section integration equation in Eq. (21), which is the other key equation in St. Venant torsion theory.

General Expression for *J*

The preceding derivations yielded the differential equation in Eq. (27) and the section integration equation in Eq. (21). Both are formulated in terms of Prandtl's stress function, φ. Hence, the challenge in St. Venant torsion is not to determine the rotation ϕ from Eq. (27), but to determine P. Once the stress function is determined, the stresses are obtained from Eq. (13). The stress function also determines the cross-sectional constant for St. Venant torsion, J. The general expression for J as a function of P is established by combining Eq. (27) and Eq. (21) into one equation of the form of Eq. (7). Specifically, substitution of T from Eq. (21) into the left-hand side of Eq. (7) and substitution $\phi_{,x}$ from Eq. (27) into the right-hand side of Eq. (7) yields:

$$\underbrace{\left(2\cdot\int_A P\cdot \mathrm{d}A\right)}_{T}=GJ\cdot\underbrace{\left(-\frac{P_{,yy}+P_{,zz}}{2\cdot G}\right)}_{\phi_{,x}} \tag{28}$$

Solving for J yields:

$$J = -\frac{4 \cdot V}{\nabla^2 P} \tag{29}$$

where V is the volume under the stress function, written as an integral in Eq. (21). Figure 4 summarizes the governing equations for St. Venant torsion.

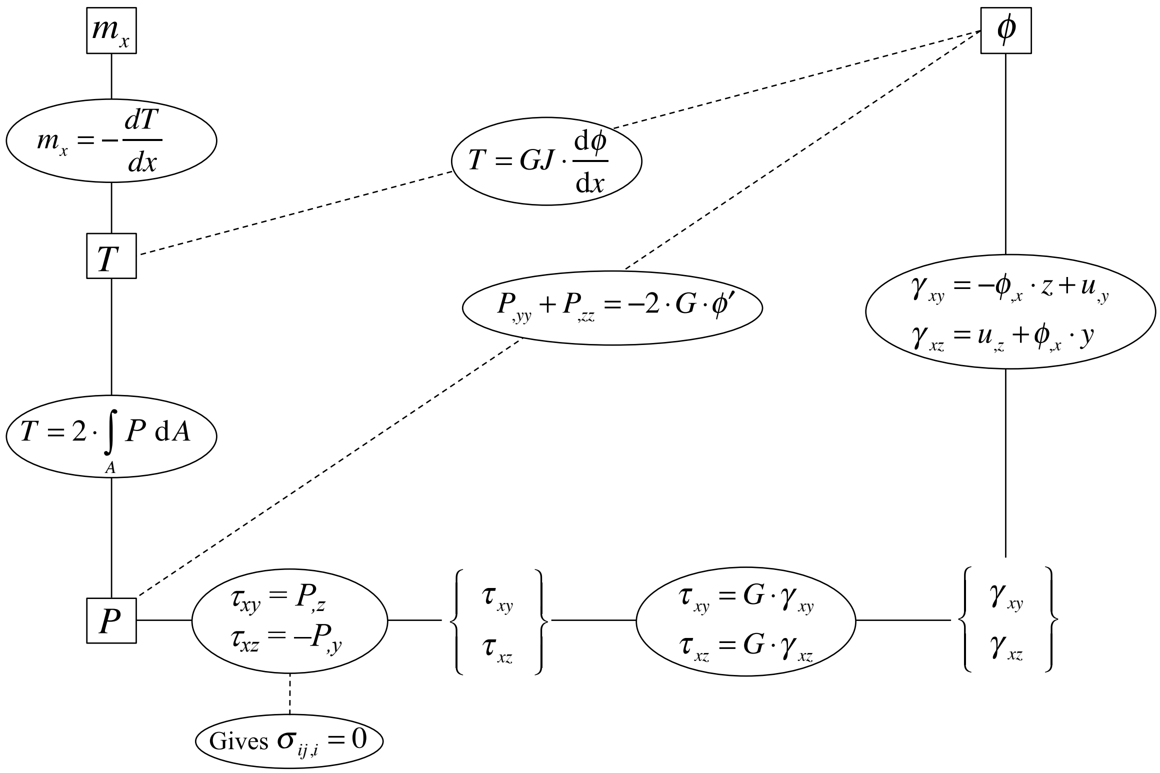

Figure 4: Governing equations for arbitrary open cross-sections.

Alternative Expression for *J*

Later in this document, Eq. (29) is employed to determine J for specific cross-section types. For completeness, an alternative formulation of Eq. (29) is presented by first integrating the differential equation in Eq. (27):

$$\int_{A_\Gamma} P_{,yy} + P_{,zz} \, dA = -2 \cdot G \cdot \phi_{,x} \int_{A_\Gamma} dA \tag{30}$$

where Γ is an arbitrary closed path in the cross-section, and A_Γ is the area within that path. Integration by parts yields

$$\int_{A_\Gamma} P_{,yy} + P_{,zz} \, dA = \oint_\Gamma \left(P_{,y} dz - P_{,z} dy \right) \tag{31}$$

Introduction of stresses from Eq. (13) yields

$$\oint_\Gamma \left(P_{,y}dz - P_{,z}dy\right) = \oint_\Gamma \left(\tau_{xz}dz + \tau_{xy}dy\right) = \oint_\Gamma \left(\tau_{xz}\frac{dz}{ds} + \tau_{xy}\frac{dy}{ds}\right)ds = \oint_\Gamma \tau_{xs}\,ds \tag{32}$$

Substitution back into Eq. (30) yields

$$\oint_\Gamma \tau_{xs}\,ds = -2\cdot G\cdot \phi'\cdot A_\Gamma \tag{33}$$

Employing this version of the differential equation to substitute into the right-hand side of Eq. (7), while still substituting the expression for T from Eq. (21) into the left-hand side of Eq. (7) yields

$$2\cdot\int_A P\cdot \mathrm{d}A = -GJ\cdot\frac{\oint_\Gamma \tau_{xs}\,ds}{2\cdot G\cdot A_\Gamma} \tag{34}$$

Solving for J yields:

$$J = -\frac{4\cdot V\cdot A_\Gamma}{\oint_\Gamma \tau_{xs}\,ds} \tag{35}$$

Thin-walled Open Cross-sections

Consider a "thin-walled" cross-section, such as a wide-flange beam and an L-shaped cross-section. In the plane of the cross-section the parts may be straight or not, but everywhere the thickness t is significantly smaller than the length of the part of the cross-section. Furthermore, there are no cells, i.e., cavities, in the cross-section. Finally, let the cross-sectional coordinate r be defined to always be perpendicular to the longitudinal coordinate, s, of any cross-section part. Then consider the stress function

$$P(r,s) = k\cdot\left(1-4\cdot\frac{r^2}{t^2}\right) \tag{36}$$

In words, this is a "quadratic pillow" with magnitude k that follows the longitudinal direction of all cross-section parts. Wherever a cross-section part ends there is a minor error; the stress function is not zero at that edge. However, Eq. (36) remains appropriate as long as t is small. The stress function in Eq. (36) is utilized to evaluate V and $\nabla^2\varphi$ in order to evaluate Eq. (29):

$$\left.\begin{aligned} V \equiv \int_A P\cdot \mathrm{d}A = \int_{-t/2}^{t/2}\int_{-b/2}^{b/2} k\left(1-4\cdot\frac{r^2}{t^2}\right)ds\,dr = \frac{2}{3}\cdot k\cdot t\cdot b \\ \nabla^2 P \equiv P_{,ss} + P_{,rr} = -\frac{8k}{t^2} \end{aligned}\right\} \quad\Rightarrow\quad J = \frac{1}{3}\cdot t^3\cdot b \tag{37}$$

Shear Stress

The stress function is

$$P(r,s) = k \cdot \left(1 - 4 \cdot \frac{r^2}{t^2}\right) \tag{38}$$

The stresses are computed from the stress function:

$$\begin{aligned} \tau_{xs} &= P_{,r} = \frac{8k}{t^2} \cdot r \\ \tau_{xr} &= -P_{,s} = 0 \end{aligned} \tag{39}$$

where k is determined from the value of the torque, T:

$$T = 2 \cdot \int_A P \cdot \mathrm{d}A = 2 \cdot V \tag{40}$$

where the volume under the stress function is

$$V = \frac{2}{3} \cdot k \cdot t \cdot b \tag{41}$$

As a result, the shear stress in the s-direction is

$$\tau_{xs} = \left(\frac{8 \cdot r}{t_i^2}\right) \cdot k = \left(\frac{8 \cdot r}{t_i^2}\right) \cdot \left(\frac{3 \cdot T}{4 \cdot t \cdot b}\right)$$

(42)

where the subscript i is introduced to identify the thickness, t_i, at the location where the stress is computed, while $t \cdot b$ in the last parenthesis is summed over the entire cross-section because it is part of the computation of the volume under the stress function. It is noted that the shear stresses, and equivalently the shear flow, "circulate" in the cross-section. The flow is parallel to the longitudinal direction of each cross-section part, with opposite direction on each side of the mid-line. It is largest at the edges of the cross-section.

Cross-sections with Several Parts

Cross-sections with several parts like the one addressed above are dealt with as follows. The total torque, T, is carried by superposition:

$$T = \sum T_i \tag{43}$$

where T_i is the torque in each part. The expression for the torque in each part of the cross-section is substituted from Eq. (7):

$$T = \sum \left(GJ_i \cdot \frac{\mathrm{d}\phi_i}{\mathrm{d}x} \right) \tag{44}$$

However, for the cross-section to retain its shape, all parts of the cross-section must rotate by the same angle ϕ, which yields:

$$T = \sum \left(GJ_i \cdot \frac{\mathrm{d}\phi}{\mathrm{d}x} \right) = \frac{\mathrm{d}\phi}{\mathrm{d}x} \cdot \sum \left(GJ_i \right) \equiv \frac{\mathrm{d}\phi}{\mathrm{d}x} \cdot GJ \tag{45}$$

Hence, it is concluded that the total torsional stiffness GJ is obtained by summing contributions GJ_i from all parts of the cross-section. Furthermore, it is inferred that the torque on each part of the cross-section is relative to the torsional stiffness of that part:

$$T_i = GJ_i \cdot \frac{d\phi_i}{dx} = \frac{GJ_i}{\sum(GJ_i)} \cdot T \tag{46}$$

That leads to the following calculation procedure for cross-sections consisting of parts that have different thicknesses, or different values of G:

1. Calculate the cross-section constant, J_i, for each cross-section part
2. Sum GJ_i over all parts to get the Saint Venant stiffness torsional stiffness for the entire cross-section
3. Calculate the fraction of the total torque carried by each part of the cross-section using Eq. (46)
4. Use the section integration equation $T=2V$ to calculate the volume underneath the stress function of each part: $V=T/2$
5. Determine the constant, k_i, in the stress function for that part
6. Substitute k_i into the formula for stress in Eq. (42)

Cross-sections with Cells

Closed cross-sections have one or more cells and carry torque better than open cross-sections. If one draws a continuous line around the circumference of the cross-section, then there will be more than one line: One will outline the external circumference and the other(s) will outline the openings (cells). When postulating a stress function under these circumstances then the required constant value around the edge may be different around the different edges. Hence, a stress function with more than one unknown is needed. In turn, more equations are needed to determine the unknowns. Continuity equations come to rescue. They express that the net axial displacement around any cell must be zero:

$$\oint_{\substack{\text{Closed}\\\text{curve}}} du = 0 \tag{47}$$

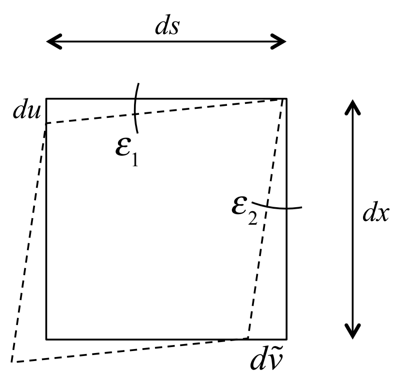

Figure 5: Contributions to total shear strain in an infinitesimal element.

To identify du, first let s denote an axis that follows the closed curve in the counterclockwise direction, and let $\tilde{v}$ denote the displacement in the s-direction. Similar to Eq. (25), and with strain contributions shown in Figure 5, kinematics yield the following equation that contains du:

$$\gamma_{xs} = \varepsilon_{xs,1} + \varepsilon_{xs,2} = \frac{du}{ds} + \frac{d\tilde{v}}{dx} \tag{48}$$

Rearranging and substituting material law ($\tau_{xs}=G\cdot\gamma_{xs}$) yields

$$\mathrm{d}u = \left(\frac{\tau_{xs}}{G} - \frac{d\tilde{v}}{dx} \right) \cdot \mathrm{d}s \tag{49}$$

As a result the continuity requirement in Eq. (47) reads

$$\oint_{\substack{\text{Closed}\\\text{curve}}} \left(\frac{\tau_{xs}}{G} - \frac{\mathrm{d}\tilde{v}}{\mathrm{d}x} \right) \mathrm{d}s = 0 \tag{50}$$

Next, the displacement $\tilde{v}$ in the s-direction is expressed in terms of the cross-section rotation, ϕ, and the distance from the centre of rotation to the tangent line of the s-axis at any location along the closed curve:

$$\tilde{v} = \phi \cdot h \tag{51}$$

The continuity equation now reads:

$$\oint_{\substack{\text{Closed}\\\text{curve}}} \left(\frac{\tau_{xs}}{G} - \frac{\mathrm{d}\phi}{\mathrm{d}x} \cdot h \right) \mathrm{d}s = \oint_{\substack{\text{Closed}\\\text{curve}}} \frac{\tau_{xs}}{G} \mathrm{d}s - \frac{\mathrm{d}\phi}{\mathrm{d}x} \cdot \oint_{\substack{\text{Closed}\\\text{curve}}} h \, \mathrm{d}s = 0 \tag{52}$$

With reference to Figure 6, the last integral is expressed in terms of the area within the closed curve, denoted A. Specifically, the product $h \cdot ds$ is twice the area of the shaded region in Figure 6, therefore:

$$\oint_{\substack{\text{Closed}\\\text{curve}}} h \cdot \mathrm{d}s \equiv 2 \cdot A \tag{53}$$

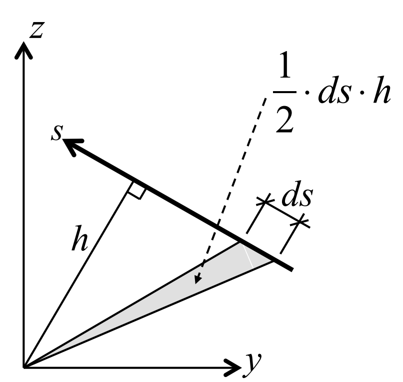

Figure 6: Evaluation of the integral of $h \cdot ds$.

As a result, the continuity equation reads:

$$\oint_{\substack{\text{Closed}\\\text{curve}}} \frac{\tau_{xs}}{G} \mathrm{d}s - \frac{\mathrm{d}\phi}{\mathrm{d}x} \cdot 2 \cdot A = 0 \tag{54}$$

By invoking Eq. (7) to express $d\phi/dx$ the final version of the continuity equation is:

$$\oint_{\substack{\text{Closed}\\\text{curve}}} \tau_{xs} \, \mathrm{d}s = \frac{T}{J} \cdot 2 \cdot A \tag{55}$$

This equation includes kinematics and material law, but not the section integration equation, and therefore serves a similar role to that of the differential equation in Eq. (27).

Thin-walled Cross-sections with One Cell

The assumed stress function has zero amplitude around the external edge, and amplitude K around the internal edge. It has a quadratic variation in between, with M denoting the height of the parabola above the 0-to-K amplitude. Letting the s-coordinate run along the mid-line, with an r-coordinate running perpendicular, the stress function reads:

$$P(s,r) = K \cdot \left(\frac{1}{2} + \frac{r}{t} \right) + M \cdot \left(1 - \left(\frac{2r}{t} \right)^2 \right) \tag{56}$$

Two equations are required to determine K and M. To this end, two continuity equations are established. One of these could be replaced by the differential equation expressed in Eq. (29), but an important point about the magnitude of M is better made with two continuity equations. The shear stress needed for these equations is:

$$\tau_{xs} = \frac{\partial P}{\partial r} = \frac{K}{t} - \frac{8Mr}{t^2} \tag{57}$$

The continuity equation in Eq. (55) expressed along the mid-line where r=0 is:

$$K \cdot \oint_{\Gamma_m} \frac{1}{t} \mathrm{d}s = \frac{T}{J} \cdot 2 \cdot A_m \tag{58}$$

The continuity equation along the inner edge where r=t/2 is:

$$K \cdot \oint_{\Gamma_i} \frac{1}{t} \mathrm{d}s - 4M \cdot \oint_{\Gamma_i} \frac{1}{t} \mathrm{d}s = \frac{T}{J} \cdot 2 \cdot A_i \tag{59}$$

Combining Eqs. (58) and (59) yields:

$$M = \frac{K}{4} \cdot \left(1 - \frac{A_i}{A_m} \cdot \frac{\oint\limits_{\Gamma_m} \frac{1}{t} \mathrm{d}s}{\oint\limits_{\Gamma_i} \frac{1}{t} \mathrm{d}s} \right) \tag{60}$$

The line integral of $1/t$ is readily determined in the fashion $L_1/t_1 + L_2/t_2 + \cdots$ where L_1 is the length of the cross-section part with thickness t_1 and so forth. The observation is now made that for thin-walled cross-sections the two last fractions are both approximately equal to unity, hence M≈0. Consequently, the stress function has only one unknown, K:

$$P(s,r) = K \cdot \left(\frac{1}{2} + \frac{r}{t} \right) \tag{61}$$

which implies that the stress in the s-direction is K/t, which in turn implies that the shear flow around the cell is K. In contrast with the derivations that produced the general expression for J in Eq. (29), the continuity equation in Eq. (58) takes the place of the differential equation in Eq. (27). The other ingredient is the section integration in Eq. (21), which for this stress function reads:

$$T = 2 \cdot \int\limits_A P \cdot \mathrm{d}A = 2 \cdot K \cdot A_m \tag{62}$$

Substitution into the continuity equation in Eq. (58) yields what is sometimes called "Bredt's formula" or "Bredt's second formula:"

$$J = \frac{4 \cdot A_m^2}{\oint\limits_{\Gamma_m} \frac{1}{t} \mathrm{d}s} \tag{63}$$

Shear Stress

Here the stress function is

$$P(s,r) = K \cdot \left(\frac{1}{2} + \frac{r}{t} \right) \tag{64}$$

To compute the shear stress for these cross-sections it is necessary to determine the value of the constant K in the stress function in terms of the applied torque, T. The section integration equation yields:

$$T = 2 \cdot \int\limits_A P \cdot \mathrm{d}A = 2 \cdot K \cdot A_m \quad \Rightarrow \quad K = \frac{T}{2 \cdot A_m} \tag{65}$$

In turn, the shear stress in the s-direction is obtained, by the definition of the stress function, by differentiating the stress function in Eq. (61):

$$\tau_{xs} = P_{,r} = \frac{K}{t} \tag{66}$$

It is here noted that the shear flow for a closed cross-section is entirely different from that of an open one. In the closed cross-section the shear stress is constant through the thickness and flows around the cell in one large loop. In fact, Eq. (66) reveals that the shear flow is constant and equal to *K* around the cell:

$$q_s = \tau_{xs} \cdot t = K \tag{67}$$

Cross-sections with Several Parts

To derive an expression for the torsional stiffness *GJ* for composition cross-sections, which was done earlier for open cross-sections, the continuity equation in Eq. (52) is revisited. However, this time *G* may vary and cannot be pulled out of the integral. Hence, the continuity equation is:

$$\oint_{\substack{\text{Closed}\\\text{curve}}} \tau_{xs}\, \mathrm{d}s = \frac{\mathrm{d}\phi}{\mathrm{d}x} \cdot \oint_{\substack{\text{Closed}\\\text{curve}}} G \cdot h\, \mathrm{d}s \tag{68}$$

This time, substitution of Eq. (7) yields:

$$\oint_{\substack{\text{Closed}\\\text{curve}}} \tau_{xs}\, \mathrm{d}s = \frac{T}{GJ} \cdot \sum \left(2A_i G_i\right) \tag{69}$$

Substitution of the stress function from Eq. (61) yields:

$$K \cdot \oint_{\substack{\text{Closed}\\\text{curve}}} \frac{1}{t} \mathrm{d}s = \frac{T}{GJ} \cdot \sum \left(2A_i G_i\right) \tag{70}$$

Again combining the continuity equation with the stress resultant equation in Eq. (62) yields

$$GJ = \frac{2 \cdot A_m}{\oint_{\Gamma_m} \frac{1}{t} \mathrm{d}s} \cdot \sum \left(2A_i G_i\right) \tag{71}$$

Thin-walled Cross-sections with Multiple Cells

Above, the stress function in Eq. (56) was suggested for a one-cell cross-section. However, it was found that for thin-walled cross-sections *M* is small. This resulted in the stress function in Eq. (61), which varies linearly from zero at the outer edge to *K* at the inner edge. As a result, *K* is the value of the shear flow around the cell and cross-sections with more than one cell may have a different shear flow around each cell. In other words, each cell is associated with the stress function in Eq. (61), but each cell has a different value of K_i, where *i* identifies each cell by a number. The solution approach is again to demand continuity around each cell, expressed earlier in Eq. (55):

$$\oint_{\substack{\text{Closed}\\\text{curve}}} \tau_{xs}\, \mathrm{d}s = \frac{T}{J} \cdot 2 \cdot A_m \tag{72}$$

where A is the area within the closed curve that is traced around the cell. Provided that $\tau_{xs}=\varphi_{,r}=K_i/t$ it is straightforward to combine Eqs. Eq. (61) and (72). However, caution must be exercised in in the computation of τ_{xs} for the wall that separates the cells. There, the shear flow from both cells contributes. Specifically, the shear flow is K_1 around Cell 1, except in the wall that is adjacent to Cell 2, where the shear flow is K_1–K_2. Similarly, the shear flow around Cell 2 is K_2, except in the wall that is adjacent to Cell 1, where the shear flow is K_2–K_1. As a result, continuity around Cell 1 according to Eq. (72) requires:

$$\oint_{\Gamma_{m,1}} \frac{K_1}{t} ds - \int_{W_{1,2}} \frac{K_2}{t} ds = \frac{T}{J} \cdot 2 \cdot A_{m,1} \tag{73}$$

where $\Gamma_{m,i}$ is the line around the entire Cell i at mid-wall, $W_{1,2}$ is line along the wall that separates Cell 1 and 2, and $A_{m,i}$ is the area within Cell i, measured from the line at mid-wall. Continuity around Cell 2 requires:

$$\oint_{\Gamma_{m,2}} \frac{K_2}{t} ds - \int_{W_{1,2}} \frac{K_1}{t} ds = \frac{T}{J} \cdot 2 \cdot A_{m,2} \tag{74}$$

The constants K_i can be pulled out of the integrals, thus the two equations (73) and (74) in the two unknowns K_1 and K_2 can be written in matrix form:

$$\begin{bmatrix} \oint\limits_{\Gamma_{m,1}} \frac{ds}{t} & -\int\limits_{W_{1,2}} \frac{ds}{t} \\ -\int\limits_{W_{1,2}} \frac{ds}{t} & \oint\limits_{\Gamma_{m,2}} \frac{ds}{t} \end{bmatrix} \begin{Bmatrix} K_1 \\ K_2 \end{Bmatrix} = \frac{T}{J} \cdot 2 \cdot \begin{Bmatrix} A_{m,1} \\ A_{m,2} \end{Bmatrix} \tag{75}$$

Inversion of the coefficient matrix yields the solution:

$$\begin{Bmatrix} K_1 \\ K_2 \end{Bmatrix} = 2 \cdot \frac{T}{J} \cdot \begin{bmatrix} \oint\limits_{\Gamma_{m,1}} \frac{ds}{t} & -\int\limits_{W_{1,2}} \frac{ds}{t} \\ -\int\limits_{W_{1,2}} \frac{ds}{t} & \oint\limits_{\Gamma_{m,2}} \frac{ds}{t} \end{bmatrix}^{-1} \begin{Bmatrix} A_{m,1} \\ A_{m,2} \end{Bmatrix} \tag{76}$$

As before, the continuity equations are combined with the section integration equation, which was provided in Eq. (62) for the stress function at hand:

$$T = 2 \cdot \int_A \varphi(y,z) dA = 2 \cdot K_1 \cdot A_{m,1} + 2 \cdot K_2 \cdot A_{m,2} = 2 \cdot \begin{Bmatrix} K_1 \\ K_2 \end{Bmatrix}^T \begin{Bmatrix} A_{m,1} \\ A_{m,2} \end{Bmatrix} \tag{77}$$

Substituting Eq. (76) into (77) and solving for J yields

$$J = 4 \cdot \left[\left[\begin{array}{cc} \oint_{\Gamma_{m,1}} \frac{ds}{t} & -\int_{W_{1,2}} \frac{ds}{t} \\ -\int_{W_{1,2}} \frac{ds}{t} & \oint_{\Gamma_{m,2}} \frac{ds}{t} \end{array} \right]^{-1} \left\{ \begin{array}{c} A_{m,1} \\ A_{m,2} \end{array} \right\} \right]^T \left\{ \begin{array}{c} A_{m,1} \\ A_{m,2} \end{array} \right\} \quad (78)$$

Shear Stress

Here the continuity equations, formulated for each cell, are gathered in a linear system of equations that yield the constants, K_i, for the stress function in each cell:

$$\left\{ \begin{array}{c} K_1 \\ K_2 \end{array} \right\} = 2 \cdot \frac{T}{J} \cdot \left[\begin{array}{cc} \oint_{\Gamma_{m,1}} \frac{ds}{t} & -\int_{W_{1,2}} \frac{ds}{t} \\ -\int_{W_{1,2}} \frac{ds}{t} & \oint_{\Gamma_{m,2}} \frac{ds}{t} \end{array} \right]^{-1} \left\{ \begin{array}{c} A_{m,1} \\ A_{m,2} \end{array} \right\} \quad (79)$$

According to Eq. (67), K_i is the shear flow around each cell. Hence, once K_i are determined from Eq. (76), the shear stress is computed by dividing K_i by the wall thickness. In walls between cells, the shear flow contributions from the adjacent cells are subtracted to yield continuous shear flow.

Arbitrary Solid Cross-sections

General cross-sections require the use of numerical methods, such as the finite element method. This is a game-changer for Prandtl's stress function because the modelling of that function would require a finite element mesh even in openings of the cross-section. Instead, the approach originally adopted by St. Venant, i.e., modelling the warping is adopted. The following finite element modelling of warping caused by torsion is addressed on Page 342 in Meek's book (1991) and on Page 533 of Kolbein Bell's finite element book (Bell 2013). The objective is the same as in the warping torsion document: to determine the omega function, $\Omega(y,z)$, over the cross-section. Physically, that function represents the axial displacement throughout the cross-section caused by a unit twist rate: $u(y,z)=\Omega(y,z)\phi'$. Now consider a cantilevered beam that is completely fixed at x=0. The cross-section rotates about the shear centre, which implies the following displacements within the cross-section due to that twist:

$$v = -(\phi' x) \cdot (z - z_{sc}) = -\phi' x \cdot z + \phi' x \cdot z_{sc} \quad (80)$$

$$w = (\phi' x) \cdot (y - y_{sc}) = \phi' x \cdot y - \phi' x \cdot y_{sc} \quad (81)$$

The second term in both equations demand attention:

$$v = \phi' x \cdot z_{sc} \quad (82)$$

$$w = -\phi' x \cdot y_{sc} \quad (83)$$

First note that they vanish if the shear centre coincides with the centroid; interestingly, they can be interpreted as the out-of-plane rotation of the cross-section if the rotation, ϕ, was applied about the centroid. Remember, if the rotation is not applied about the shear centre then the beam would bend, and conversely; if a point load is not applied at the shear centre then the beam would rotate. To appreciate this, suppose the cantilevered beam is subjected to a rotation about the z-axis at the free end. This implies that the beam will bend, with the amount of displacement increasing proportional to the distance from the fixed end:

$$v = \theta_z' \cdot x \tag{84}$$

where θ_z' is the change in rotation per unit length. Similarly, an applied rotation at the free end about the y-axis implies the following downwards displacement:

$$w = -\theta_y' \cdot x \tag{85}$$

A comparison of Eqs. (82) and (84) yields

$$\theta_z' = \phi' \cdot z_{sc} \tag{86}$$

Similarly, a comparison of Eqs. (83) and (85) yields

$$\theta_y' = \phi' y_{sc} \tag{87}$$

This means that rotation about the y- and z-axes has been linked to the distance from the centroid to the shear centre. This is important, because it quantifies the bending that would occur if a torsional twist is applied *away from* the shear centre. This will help determine y_{sc} and z_{sc}. To that end, consider the total axial displacement in the cross-section due to twist and bending:

$$u(y,z) = \phi' \cdot \Omega(y,z) + \theta_y' \cdot z - \theta_z' \cdot y \tag{88}$$

where ψ is the warping of the cross-section caused by torsional twist. Substitution of Eqs. (86) and (87) into Eq. (88) yields

$$u(y,z) = \phi' \cdot \Omega(y,z) + \phi' \cdot y_{sc} \cdot z - \phi' \cdot z_{sc} \cdot y \tag{89}$$

Note that all terms in Eq. (89) are related to twist, not bending. The following two equivalent conditions can now be applied to determine y_{sc} and z_{sc}: If pure bending is applied it should not cause torsional twist; or, if twist is applied then it should not cause bending deformation. Following Meek (1991) and Bell (2013) the condition is enforced using the principle of virtual displacements: a real stress distribution in the cross-section, $\sigma(y,z)$, caused by real bending moment applied about the y-axis should produce zero work if it is acting over $\delta w(y,z)$, i.e., the virtual displacement given by Eq. (89). Because the stress distribution due to bending about the y-axis is some constant, c, multiplied by z the virtual work is

$$
\begin{aligned}
\delta W_{int} &= \int_A c \cdot z \cdot u(y,z) dA = \int_A c \cdot z \cdot \left(\phi' \cdot \Omega(y,z) + \phi' \cdot y_{sc} \cdot z - \phi' \cdot z_{sc} \cdot y \right) dA \\
&= \int_A c \cdot z \cdot \left(\phi' \cdot \Omega(y,z) + \phi' \cdot y_{sc} \cdot z - \phi' \cdot z_{sc} \cdot y \right) dA \\
&= \int_A \left(c \cdot z \cdot \phi' \cdot \Omega(y,z) + c \cdot z \cdot \phi' \cdot y_{sc} \cdot z - c \cdot z \cdot \phi' \cdot z_{sc} \cdot y \right) dA \\
&= c \cdot \phi' \cdot \int_A z \cdot \Omega(y,z) \, dA + c \cdot \phi' \cdot y_{sc} \cdot \int_A z^2 \, dA - c \cdot \phi' \cdot z_{sc} \cdot \int_A y \cdot z \, dA \\
&= c \cdot \phi' \cdot \int_A z \cdot \Omega(y,z) \, dA + c \cdot \phi' \cdot y_{sc} \cdot I_y - c \cdot \phi' \cdot z_{sc} \cdot I_{yz}
\end{aligned}
\tag{90}
$$

Equating the virtual work to zero yields

$$
y_{sc} = z_{sc} \cdot \frac{I_{yz}}{I_y} - \frac{\int_A z \cdot \Omega(y,z) \, dA}{I_y} \tag{91}
$$

Applying the same condition to bending about the z-axis yields

$$
z_{sc} = y_{sc} \cdot \frac{I_{yz}}{I_z} + \frac{\int_A y \cdot \Omega(y,z) \, dA}{I_z} \tag{92}
$$

Eqs. (91) and (92) are solved simultaneously for y_{sc} and z_{sc}. If the y- and z-axes are the principal axes then I_{yz}=0 and the answers are

$$
y_{sc} = -\frac{\int_A z \cdot \Omega(y,z) \, dA}{I_y} \quad \text{and} \quad z_{sc} = \frac{\int_A y \cdot \Omega(y,z) \, dA}{I_z} \tag{93}
$$

Because $u(y,z)=\Omega(y,z)\phi'$, the function Ω represents the warping due to a unit rate of twist: $\phi'=1$. The determination of that warping is a computational problem that can be solved using the finite element method. Following Meek (1991) and Bell (2013) the kinematic conditions in Eq. (22) are first written in vector form:

$$
\boldsymbol{\varepsilon} = \begin{Bmatrix} \gamma_{xy} \\ \gamma_{xz} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial y} \\ \dfrac{\partial u}{\partial z} \end{Bmatrix} + \phi' \cdot \begin{Bmatrix} -z \\ y \end{Bmatrix} \tag{94}
$$

where the following important observation is made: Only the first term in the right-hand side depend on the warping in the cross-section. In fact, the second term can be interpreted as the initial strain caused by the twist and this term will act as the “loading” that causes the warping. To that end, the following notation is employed, once the substitution $u=\Omega\phi'$ is made and a unit twist rate, $\phi'=1$, is enforced:

$$\boldsymbol{\varepsilon} = \begin{Bmatrix} \dfrac{\partial\Omega(y,z)}{\partial y} \\ \dfrac{\partial\Omega(y,z)}{\partial z} \end{Bmatrix} + \begin{Bmatrix} -z \\ y \end{Bmatrix} \equiv \boldsymbol{\varepsilon}_\omega + \boldsymbol{\varepsilon}_0 \tag{95}$$

The material laws in Eq. (25) are also written on matrix form:

$$\boldsymbol{\sigma} = \begin{Bmatrix} \tau_{xy} \\ \tau_{xz} \end{Bmatrix} = \begin{bmatrix} G & 0 \\ 0 & G \end{bmatrix} \begin{Bmatrix} \gamma_{xy} \\ \gamma_{xz} \end{Bmatrix} = \mathbf{C}\boldsymbol{\varepsilon} \tag{96}$$

These stress and strain expressions are now suitable for entry into a finite element analysis of St. Venant torsion, which is addressed in another document on this website.

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