Euler-Bernoulli Beams

The Euler-Bernoulli beam theory was established around 1750 with contributions from Leonard Euler and Daniel Bernoulli. Bernoulli provided an expression for the strain energy in beam bending, from which Euler derived and solved the differential equation. That work built on earlier developments by Jacob Bernoulli. However, the beam problem had been addressed even earlier. Galileo attempted one formulation that aimed at determining the capacity of beams in bending, but misplaced the neutral axis. Earlier, Leonardo da Vinci also seems to have addressed the problem of beam bending. The two key assumptions in the Euler-Bernoulli beam theory are:

- The material is linear elastic according to Hooke's law
- Plane sections remain plane and perpendicular to the neutral axis

The latter is referred to as Navier's hypothesis. In contrast, Timoshenko beam theory, which is covered in another document, relaxes the assumption that the sections remain perpendicular to the neutral axis, thus including shear deformation. In the following, the governing equations are established, followed by the formulation and solution of the differential equation. Thereafter, a section cross-section analysis describes the computation of stresses and cross-section constants. The starting point is 2D beam bending with the following sign conventions:

- The z-axis is increases upwards
- Displacement w is positive in the direction of the z-axis
- Distributed load q_z is positive in the direction of the z-axis
- Bending moment that imposes tension at the bottom is positive
- Clockwise shear force is positive
- Counter-clockwise rotation θ is positive, thus it can be interpreted as the slope of the deformed beam element
- Tensile stresses and strains are positive, compression is negative

Equilibrium

The equilibrium equations are obtained by considering equilibrium in the x-direction for the infinitesimal beam element in Figure 1. The notation q_z is employed for distributed load in the upwards direction, i.e., in the positive z-direction. Vertical equilibrium yields:

$$q_z = \frac{dV}{dx} \tag{1}$$

Moment equilibrium about the rightmost edge yields:

$$V = \frac{dM}{dx} \tag{2}$$

In Eq. (2) the "second-order terms" that contain dx^2 are neglected.

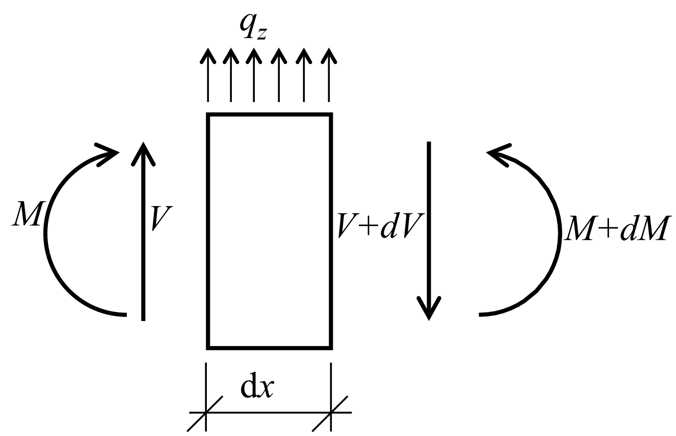

Figure 1: Equilibrium for infinitesimally small beam element.

Section Integration

Integration of axial stresses over the cross-section:

$$M = \int_A -\sigma \cdot z \ \mathrm{d}A \tag{3}$$

where the minus sign appears because it is compressive (negative) stresses in the positive z-axis domain that gives a positive bending moment, i.e., bending moment with tension at the bottom. Figure 2 is intended to explain this further.

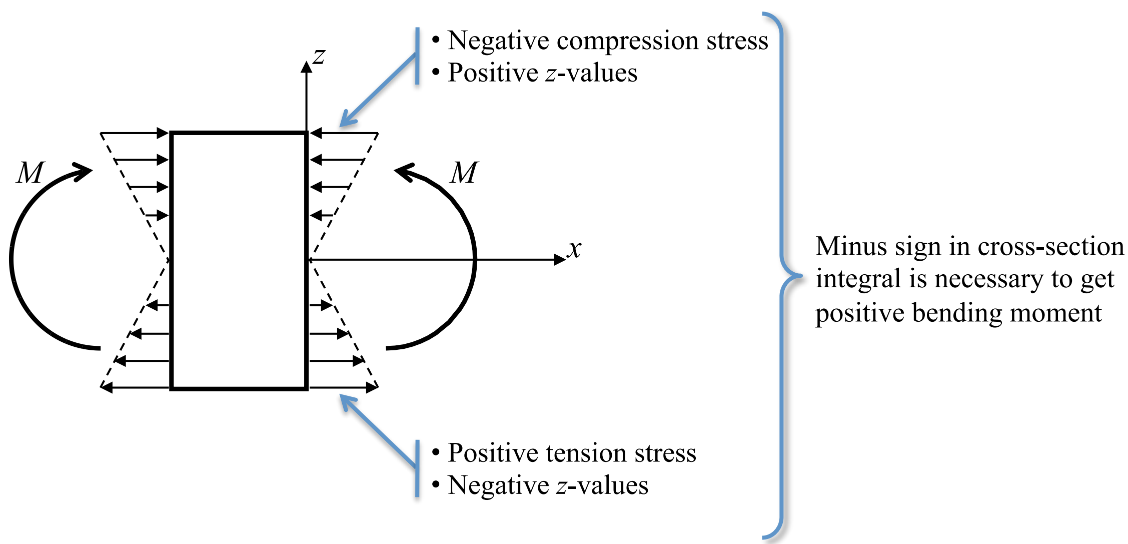

Figure 2: The reason for the minus sign in Eq. (3).

Material Law

The material law throughout linear elastic theory is Hooke's law:

$$\sigma = E \cdot \varepsilon \tag{4}$$

In the context of two-dimensional theory of elasticity, the use of Eq. (4) implies a "plane stress" material law. It implies that there is zero stress, i.e., air on the sides of the beam. The alternative "plane strain" version of the two-dimensional Hooke's law is more appropriate in cases where the beam is only a strip of a long rectangular plate that is supported along the two long edges. In that case the strain is restrained in the y-direction:

$$\varepsilon_{yy} = \frac{\sigma_{yy}}{E} - \nu \cdot \frac{\sigma_{xx}}{E} = 0 \qquad \Rightarrow \qquad \sigma_{yy} = \nu \cdot \sigma_{xx} \tag{5}$$

Which, substituted into the material law in the x-direction yields:

$$\varepsilon_{xx} = \frac{\sigma_{xx}}{E} - \nu \frac{\sigma_{yy}}{E} = \frac{\sigma_{xx}}{E} - \nu \frac{(\nu \cdot \sigma_{xx})}{E} = \frac{\sigma_{xx}}{E}(1-\nu^2) \qquad \Rightarrow \qquad \sigma_{xx} = \frac{E}{1-\nu^2} \cdot \varepsilon_{xx} \tag{6}$$

All the derivations and results in the following are based on the material law $\sigma_{xx}=E\cdot\varepsilon_{xx}$ from Eq. (4). However, the plain strain version is easily introduced by replacing the Young's modulus, E, in any equation by $E/(1-\nu^2)$.

Kinematics

The relationship between the axial strain and the transversal displacement of a beam element is sought. It is first recognized that bending deformation essentially implies shortening and lengthening of "fibres" in the cross-section. Fibres on the tension side elongate, while fibres on the compression side shorten.

The starting point for the considerations is to link the axial strain to the change of length of the imaginary fibres that the cross-section is made up of. The same consideration as in kinematics of truss members, namely that strain is "elongation divided by original length" yields:

$$\varepsilon = \frac{du}{dx} \tag{7}$$

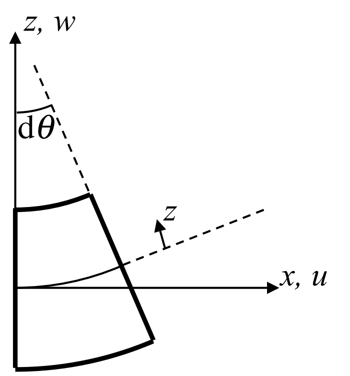

Figure 3: Navier's hypothesis for beam bending.

Next, the axial displacement u is related to the rotation of the cross-section. In particular, consider the infinitesimal counter-clockwise rotation $d\theta$ of the infinitesimally short beam element in Figure 3. In passing, it is noted that $d\theta$ is equal to the curvature, κ. Under the assumption that plane sections remain plane and perpendicular to the neutral axis during deformation, each fibre in the cross-section change length proportional to its distance from the neutral axis.

The amount of shortening or elongation depends upon the rotation of the cross-section. A geometrical consideration of to Figure 3 shows that the shortening and lengthening, i.e., axial displacement, of each infinitesimally short fibre is

$$du = -\mathrm{d}\theta \cdot z \tag{8}$$

Finally, the rotation θ is related to the transversal displacement. For this purpose, consider two points on a beam that is dx apart, as shown in Figure 4. The relative displacement is dw, which is positive upwards. Consequently, a geometrical consideration of Figure 4 shows that:

$$\tan(\theta) = \frac{dw}{dx} \approx \theta \tag{9}$$

where the equation is simplified by assuming that the deformations are sufficiently small so that $\tan(\theta) \approx \theta$.

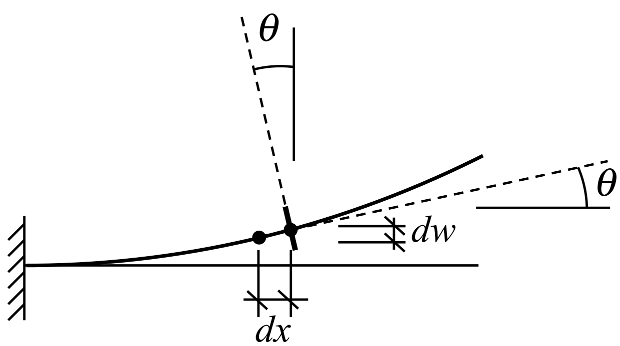

Figure 4: Rotation of the cross-section of a beam element.

Combination of the previous equations yields the following kinematic equation for beam members:

$$\varepsilon = \frac{du}{dx} = -\frac{d\theta}{dx} \cdot z = -\frac{d^2w}{dx^2} \cdot z \tag{10}$$

This expression implies an approximation of the exact curvature of the beam. Mathematically, curvature is defined as

$$\kappa \equiv \frac{1}{R} \tag{11}$$

where R is the radius of curvature of the beam. In the Euler-Bernoulli beam theory that is presented here, the curvature is approximated by

$$\kappa \approx \frac{d\theta}{dx} \approx \frac{d^2w}{dx^2} \tag{12}$$

Notice that there are two approximation signs. The first alludes to the fact that differentiation is carried out with respect to the x-axis. Unless the deformations are negligible this is inaccurate; differentiation should be carried out with respect to the "s-axis" that follows the curving beam axis. The second approximation is due to Eq. (9). From that equation it is observed that the accurate expression for θ is:

$$\theta = \tan^{-1}\left(\frac{dw}{dx}\right) \tag{13}$$

If this expression was utilized in the derivations above then the differentiation of the inverse tan-function yields

$$\kappa \approx \frac{d\theta}{dx} = \frac{\left(\dfrac{d^2w}{dx^2}\right)}{\left(1+\left(\dfrac{dw}{dx}\right)^2\right)} \tag{14}$$

which reduces to the expression in Eq. (12) when the slope dw/dx is small. However, the curvature expression in Eq. (14) is still approximate because the differentiation is carried out with respect to the x-axis and not the beam axis. From mathematics, the exact curvature expression is:

$$\kappa = \frac{\left(\dfrac{d^2w}{dx^2}\right)}{\left(1+\left(\dfrac{dw}{dx}\right)^2\right)^{\frac{3}{2}}} \tag{15}$$

Differential Equation

The governing equations for beam bending, namely equilibrium, section integration, material law, and kinematics are summarized in Figure 5. The differential equation is obtained by combining them as follows:

$$\begin{aligned} q_z &= \frac{dV}{dx} = \frac{d^2M}{dx^2} = -\frac{d^2}{dx^2}\int_A \sigma \cdot z \;\, \mathrm{d}A \\ &= -\frac{d^2}{dx^2}\int_A E\cdot\varepsilon\cdot z \;\, \mathrm{d}A = \frac{d^2}{dx^2}\int_A E\cdot\frac{d^2w}{dx^2}\cdot z^2 \;\, \mathrm{d}A \\ &= EI_y\frac{d^4w}{dx^4} \end{aligned} \tag{16}$$

where the modulus of elasticity is assumed constant over the cross-section and the moment of inertia is defined:

$$I_y = \int_A z^2 \;\; \mathrm{d}A \tag{17}$$

In Eq. (16) it is assumed that the cross-section is homogeneous so that E is constant. For composite cross-sections this assumption is invalid, and the revised version of Eq. (16) is

$$q_z = \frac{d^4 w}{dx^4} \int_A E \cdot z^2 \ \mathrm{d}A \tag{18}$$

One approach to retain the original definition of I is to first select a reference-value of the Young's modulus and assume that all parts of the cross-section has that E-value. Next, the width of each part of the cross-section is modified if its Young's modulus is different from the reference value. The change in width is proportional to the difference in E-value. E.g., if E is twice the reference value then the width should be doubled. If this procedure is followed then Eq. (17) remains valid for the determination of I.

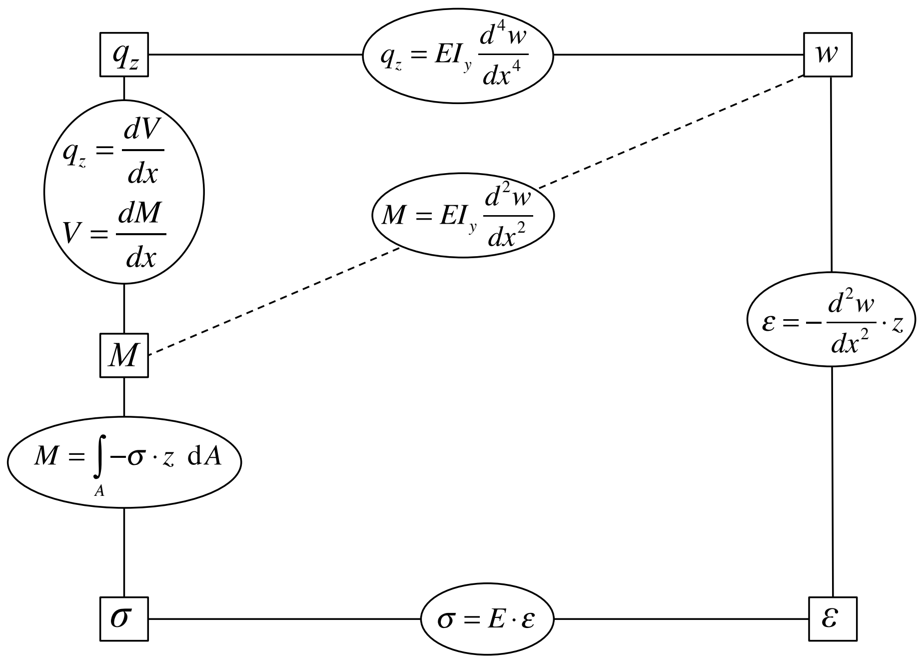

Figure 5: Governing equations in Euler-Bernoulli beam theory.

Although solving the differential equation for beam bending is rarely done in everyday engineering practice, it is instructive to study its solution for simple reference cases. In particular, the solution of the differential equation is the starting point for the selection of shape functions in the finite element method. Those shape functions are often approximate, while the solution of the differential equation reveals the exact shape when the member deforms. The general solution of the differential equation reveals whether the finite element shape functions are exact or not. For beam members, the general solution to the differential equation is obtained by integrating four times, here assuming that the load, q_z, is uniformly distributed along the beam:

$$w(x) = \frac{1}{24} \cdot \frac{q_z}{EI_y} \cdot x^4 + C_1 \cdot x^3 + C_2 \cdot x^2 + C_3 \cdot x + C_4 \tag{19}$$

It is observed that under uniform load the displaced shape of a beam is a fourth-order polynomial. Without distributed load the displaced shape is a third-order polynomial. To obtain the solution for a specific beam problem it is necessary to specify boundary conditions. To prescribe a rotation, shear force, or bending moment, the following equations are useful, obtained by combining the governing equations established earlier:

$$\theta = \frac{dw}{dx} \tag{20}$$

$$M = EI_y \frac{d^2w}{dx^2} \tag{21}$$

$$V = EI_y \frac{d^3w}{dx^3} \tag{22}$$

Cross-section Constants

Centroid

The centroid, whose coordinates are denoted y_o and z_o, is the location about which the first moments of area must be zero. Throughout this document the y- and z-axes are assumed to originate from the centroid. For the sake of these derivations, suppose another coordinate system $(\tilde{y},\tilde{z})$ has its origin elsewhere. That means the condition of zero first moments of area read

$$\begin{aligned} \int_A y\,dA &= \int_A (\tilde{y}-y_o)\,dA = \int_A \tilde{y}\,dA - \int_A y_o\,dA = \int_A \tilde{y}\,dA - y_o \cdot A = 0 \\ \int_A z\,dA &= \int_A (\tilde{z}-z_o)\,dA = \int_A \tilde{z}\,dA - \int_A z_o\,dA = \int_A \tilde{z}\,dA - z_o \cdot A = 0 \end{aligned} \tag{23}$$

Solving for the centroid coordinates y_o and z_o yields:

$$\begin{aligned} y_o &= \frac{\int_A \tilde{y}\,dA}{A} \\ z_o &= \frac{\int_A \tilde{z}\,dA}{A} \end{aligned} \tag{24}$$

In words, the location of the neutral axis is obtained by summing "area multiplied by distance" for all parts of the cross-section, and then dividing the sum by the total area. For homogeneous cross-sections the neutral axis along the beam coincides with the geometrical centroid of the cross-section. For composite cross-sections, i.e., cross-sections composed of different materials, it is possible to scale the area of each cross-section part about its local centroid. The scaling is proportional to the E-value relative to a reference value. Subsequently, the centroid of the scaled cross-section is determined, which is the sought neutral axis.

Moment of Inertia

The cross-section of any beam member has two moments of inertia, I_y and I_z. If y and z are the principal axes of the cross-section, then these two moments of inertia are the largest and smallest among all possible orientations of the orthogonal y and z axes. The subscripts on I_y and I_z indicate the axis about which the cross-section rotates under bending. As a result, their definitions are:

$$\begin{aligned} I_y &= \int_A z^2 \quad \mathrm{d}A \\ I_z &= \int_A y^2 \quad \mathrm{d}A \end{aligned} \tag{25}$$

For general cross-sections it is often convenient to divide the cross-section into parts, each with local moment of inertia denoted by I_i about its local centroid. The contributions to the global moment of inertia from each part, about the global centroid of the cross-section, are summed in accordance with the parallel axis theorem, sometimes referred to as "Steiner's sats:"

$$\begin{aligned} I_y &= \sum \left(I_{y,i} + z_i^2 \cdot A_i \right) \\ I_z &= \sum \left(I_{z,i} + y_i^2 \cdot A_i \right) \end{aligned} \tag{26}$$

where y_i and z_i are distances from the centroid of the entire cross-section, i.e., the neutral axis along the beam, to the centroid of the part. In cases where I_y and I_z about the principal axes are sought, but y and z are not the principal axes, $I_{y,principal}$ and $I_{z,principal}$ can either be calculated from scratch once the orientation, $\theta_{principal}$, of the principal axes are determined, or they can be determined by transformation of the original moments of inertia (Megson 2020):

$$\begin{aligned} I_{y,principal} &= \frac{I_y + I_z}{2} + \frac{1}{2} \cdot \sqrt{\left(\left(I_z - I_y \right)^2 + 4 \cdot I_{yz}^2 \right)} \\ I_{z,principal} &= \frac{I_y + I_z}{2} - \frac{1}{2} \cdot \sqrt{\left(\left(I_z - I_y \right)^2 + 4 \cdot I_{yz}^2 \right)} \end{aligned} \tag{27}$$

Product of Inertia

The product of inertia, $I_{yz}=I_{zy}$, is a quantity that is non-zero only for cross-sections without any axis of symmetry. If a symmetry axis exists then it is a principal axis, and thus both principal axes are immediately known. The product of inertia should not be confused with the polar moment of inertia, which is related to torsion. The product of inertia appears in the derivation of the governing equations for bi-axial bending, where it is defined as:

$$I_{yz} = \int_A y \cdot z \ \mathrm{d}A \tag{28}$$

The coordinates y and z in Eq. (28) represent distances from the neutral axis, which is a point in the cross-section, to the area elements. For cross-sections with a symmetry axis, I_{yz}=0. The parallel axis theorem, i.e., Steiner's formula, for products of inertia is:

$$I_{yz} = \sum \left(I_{yz,i} + y_i \cdot z_i \cdot A_i \right) \tag{29}$$

where y_i and z_i are distances from the global centroid to the local centroid. For example, the value of I_{yz} for a triangular cross-section with dimensions b and h is

$$I_{yz} = \frac{b^2 h^2}{72} \tag{30}$$

Principal Axes

The identification of the principal axes of a cross-section has several benefits; it simplifies the equations for stress and strain and it implies that I_y and I_z are the greatest and smallest moments of inertia of the cross-section. The product of inertia is employed to determine the principal axes; once I_{yz} is determined the following conclusions can be made in regards to the principal axes:

- If I_{yz}=0 then y and z are the principal axes
- If I_{yz}≠0 then y and z are NOT the principal axes

The orientation of the principal axes relative to the original axes is

$$\theta_{principal} = \frac{1}{2} \cdot \arctan\left(2 \cdot \frac{I_{yz}}{I_y - I_z}\right) \tag{31}$$

If I_y=I_z then that formula cannot be evaluated, but then only two options exist: The first is that I_{yz}=0, in which case θ_{axes}=0; the only other possibility is that θ_{axes}=±45°.

Axial Stress

For uniaxial bending, a convenient approach for obtaining the axial stress in terms of the bending moment is to combine material law and kinematics equations, which yields:

$$\sigma = -E \cdot \frac{d^2 w}{dx^2} \cdot z \tag{32}$$

Then substitute the differential equation without equilibrium equations to obtain:

$$\sigma = -\frac{M}{I} \cdot z \tag{33}$$

The minus sign means that a positive bending moment, i.e., tension at the bottom, correctly yields negative stresses at the top, i.e., compression, where z is positive. This is the same reason that was given for the minus sign in Eq. (33), which also correctly gives positive tension stresses at the bottom when a positive moment acts on the cross-section.

For biaxial bending, Eq. (33) is valid for bending about both the y and z axes, as long as they are the principal axes. If they are not the principal axes but their orientation, ϕ, relative to the original axes is known, then it is possible to develop formulas for stress in terms of ϕ and the original axes. These formulas will be more complicated than the fundamental ones because they include the product of inertia, I_{yz}. The starting point is the expression for strain at an arbitrary location in the cross-section:

$$\varepsilon = \varepsilon_o - \frac{d^2 v}{dx^2} \cdot y - \frac{d^2 w}{dx^2} \cdot z \tag{34}$$

The material law yields the preliminary expression for stress:

$$\sigma = E \cdot \varepsilon_o - E \cdot \frac{d^2 v}{dx^2} \cdot y - E \cdot \frac{d^2 w}{dx^2} \cdot z \tag{35}$$

Integration of axial stress yields the bending moments M_y and M_z about the y and z axes, respectively. First consider the bending moment about the y-axis:

$$M_y = -\int_A \sigma \cdot z \ \mathrm{d}A = -E \cdot \varepsilon_o \cdot \int_A z \ \mathrm{d}A + E \cdot \frac{d^2 v}{dx^2} \cdot \int_A y \cdot z \ \mathrm{d}A + E \cdot \frac{d^2 w}{dx^2} \cdot \int_A z^2 \ \mathrm{d}A \tag{36}$$

The first integral vanishes because z originates at the neutral axis, while the last term is the ordinary bending moment. As a result, Eq. (36) and its counterpart for M_z can be written as:

$$\begin{aligned} M_y &= -EI_{yz} \cdot \frac{d^2 v}{dx^2} - EI_y \cdot \frac{d^2 w}{dx^2} \\ M_z &= EI_{yz} \cdot \frac{d^2 w}{dx^2} + EI_z \cdot \frac{d^2 v}{dx^2} \end{aligned} \tag{37}$$

where the product of inertia, I_{yz}, has been defined as:

$$I_{yz} = \int_A y \cdot z \ \mathrm{d}A \tag{38}$$

Similarly, I_{yz} appears in the expression for M_z. Eq. (37) represents kinematics, material law, and section integration. In contrast, Eq. (35) represents only kinematics and material law. Thus, combination of Eq. (35) and Eq. (37) facilitates the isolation of the section integration relationship, namely the relationship between axial stress and bending moment. This is done by solving for the curvatures in Eq. (37) and substituting them into Eq. (35). Because the axial strain is N/EA the result is

$$\sigma = \frac{N}{A} + \frac{M_y}{\left(I_y \cdot I_z - I_{yz}^2\right)} \cdot \left(I_z \cdot z - I_{yz} \cdot y\right) - \frac{M_z}{\left(I_y \cdot I_z - I_{yz}^2\right)} \cdot \left(I_y \cdot y - I_{yz} \cdot z\right) \tag{39}$$

Neutral Axis

The line in the cross-section along which the axial stress is zero is here sought, assuming zero axial force, N. In uni-axial bending this task is trivial; the answer is the axis around which the bending occurs. Thus, this section addresses the problem of bi-axial bending. The location of the sought axis depends on the loading, and consider first the case of a moment, M, applied at a clockwise positive angle, θ, relative to the horizontal y-axis. The decomposition of M yields the following moments about each principal axis:

$$\begin{aligned} M_y &= M \cdot \cos(\theta) \\ M_z &= M \cdot \sin(\theta) \end{aligned} \tag{40}$$

Because y and z are the principal axes, the axial stress in the cross-section is found by the following formula from fundamental beam theory:

$$\sigma = \frac{M_y}{I_y} \cdot z + \frac{M_z}{I_z} \cdot y \tag{41}$$

In passing, it is noted that the value of θ that would yield the largest axial stress at a particular location (y, z) can here be determined. The sought neutral axis is characterized by σ=0, which according to Eqs. (40) and (41) yields the following condition:

$$\frac{M \cdot \cos(\theta)}{I_y} \cdot z + \frac{M \cdot \sin(\theta)}{I_z} \cdot y = 0 \tag{42}$$

Solved for z, Eq. (42) yields:

$$z = -\frac{I_y}{I_z} \cdot \tan(\theta) \cdot y \tag{43}$$

which is the sought result. From Eq. (43), the angle, ψ, between the y-axis and the neutral axis is solved from the equation

$$\tan(\psi) = \frac{z}{y} = -\frac{I_y}{I_z} \cdot \tan(\theta) \tag{44}$$

Shear Stress

When approaching shear stresses in bending, an anomaly in Euler-Bernoulli beam theory is observed. The theory assumes that plane sections remain plane and perpendicular to the neutral axis. In other words, the only strain that takes place is the axial shortening or elongation of the fibres in the cross-section. Effectively, this prevents shear strain. With no shear strain there is no shear stress, which adds up to zero shear force. In other words, shear strain and shear force are not part of the Euler-Bernoulli beam theory. This is an anomaly, because shear force will develop even in simple beams that are subjected to transversal load. The anomaly is addressed by recovering the shear force by equilibrium, once the bending moment is computed. This is seen in the simple beam theory, where the shear force is equal to the derivative of the bending moment; this is the equilibrium equation that recovers the shear force.

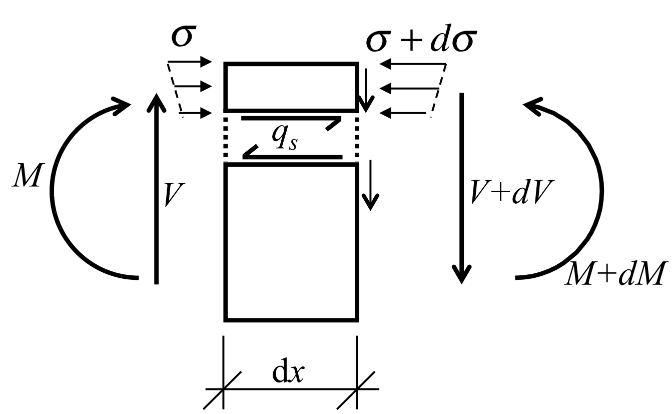

Figure 6: Shear flow by equilibrium of infinitesimal beam element.

Equilibrium considerations will also be employed in this document, in order to determine the shear stress and "shear flow" at any point in the cross-section. Another document on Timoshenko beam theory describes an approach to extend Euler-Bernoulli beam theory to include shear deformation in the beam deflection. To obtain the most popular expression for the shear stress, τ, in terms of the shear force, V, consider the infinitesimally short beam element in Figure 6. Furthermore, consider a "cut" in the cross-section and let q_s denote the "shear flow" at that location. The shear flow, q_s, is the force per unit length of the beam that ensures equilibrium with the axial stresses, which are greater on one side than the other due to dM:

$$q_s \cdot dx = \int_{A_s} d\sigma \, dA = \int_{A_s} \frac{dM}{I} \cdot z \, dA \tag{45}$$

where A_s is the cross-sectional area outside the cut. Given $V=dM/dx$ this yields

$$q_s = \frac{V}{I} \cdot Q \tag{46}$$

where the first moment of area, Q, has been defined as

$$Q = \int_{A_s} z \, dA \tag{47}$$

To ease the evaluation of Q in practice, the cross-section is often discretized into several parts with area A_i and distance z_i from the neutral axis to the centroid of the part. Then

$$Q = \sum_{i=1}^{N} z_i A_i \tag{48}$$

where N is the number of cross-section parts. Once Q is determined at a particular location, the shear stress is calculated by distributing the shear flow over the thickness, t, of the cross-section at the particular location:

$$\tau = \frac{V \cdot Q}{I \cdot t} \tag{49}$$

It is noted that Eqs. (46) and (49) are not necessarily exact. Both are approximations unless the beam is prismatic, i.e., straight along the x-axis with a uniform cross-section along the x-axis. Even for prismatic beams Eq. (49) may not be exact because the shear stresses may not be uniformly distributed over the thickness, t. Consider a shear force applied in the z-direction. In that case, Eq. (49) is an approximation unless the edges of the cross-section are parallel with the z-axis, so that the shear stresses align with the z-axis. In other words, the cross-section for which Eq. (49) is most accurate is rectangular, high, and thin. In contrast, it is more difficult to determine the shear stress distribution within a triangular cross-section. That being said, it is often possible to apply Eq. (49) as an approximation, and it is pretty good even for circular and triangular cross-sections (Timoshenko and Goodier 1969). For example, for a solid circular cross-section the edges are locally parallel to the z-axis near the centroid. Eq. (49) can therefore be used as an approximation to determine the shear stress at that location in the cross-section. Another type of cross-sections where Eq. (49) is utilized is "thin-walled" cross-sections. For such problems, a new s-axis is defined, which follows the contour of the cross-section. It is assumed that the shear flow and shear stress follow that direction, thus satisfying the aforementioned requirement. With that assumption, Eq. (49) can be applied without modifications. For thin-walled cross-sections that are "closed," i.e., consisting of one or more "cells," the shear stress cannot be solved by equilibrium alone, i.e., they are statically indeterminate, as discussed next.

Closed Thin-walled Cross-sections

The determination of shear stress and shear centre for closed cross-sections, i.e., cross-sections with cells, can be carried out in two ways. In the approach addressed first, the shear flow is determined before the shear centre. This approach explicitly recognizes that the calculation of shear flow in a closed cross-section is a "statically indeterminate" problem. In other words, equilibrium equations alone are insufficient to determine the sought forces. Each cell is associated with one redundant. Similar to the flexibility method in fundamental structural analysis, the solution approach involves removing the capacity of the structure to carry the sought forces, i.e., to introduce "cuts," and then to enforce compatibility equations that are solved for the unknown forces. In the following, a thin-walled cross-section with one cell is considered, and one cut is introduced to make it an open cross-section. At the cut there will develop a discrepancy in u-displacement at each side of the cut. The compatibility equation requires this displacement to be zero:

$$u = \oint du = 0 \tag{50}$$

where du are the infinitesimal contributions that are integrated along the s-axis, which runs through the middle thickness along the cross-section.

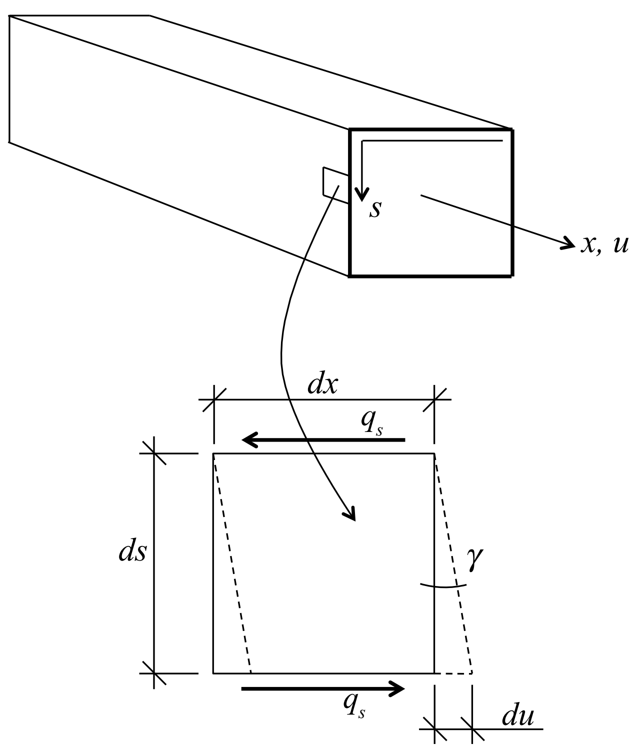

Figure 7: Kinematic relationship between *du* and shear strain *γ*.

Figure 7 shows an infinitesimal part of the cross-section of a beam element, seen from the side of the beam. The figure illustrates the kinematic relationship between the shear strain, γ, and the sought quantity du:

$$\gamma = \frac{du}{ds} \tag{51}$$

It is noted that generally there would be another contribution to the shear strain, namely $(d\phi/dx)$(h), where ϕ is the rotation of the cross-section and h is the distance from the neutral axis to the tangent of the cross-section part. However, it is understood that the rotation of the cross-section must be zero if the shear force acts through the shear centre. Thus $d\phi/dx=0$. In short, the change in axial displacement at two points located a distance ds apart is, from kinematics:

$$du = \gamma \cdot ds \tag{52}$$

Material law provides the following expression for the shear strain in terms of the shear stress and thus the shear flow:

$$\gamma = \frac{\tau}{G} = \frac{q_s}{G \cdot t} \tag{53}$$

where $G=E/(2(1+\nu))$ is the shear modulus and t is the thickness of the cross-section-wall, which may vary around the circumference. Substitution of Eq. (53) into Eq. (52) yields

$$du = \frac{q_s}{G \cdot t} \cdot ds \tag{54}$$

Integration around the cell yields the total gap opening at the cut, which is enforced to zero by compatibility:

$$u = \oint \frac{q_s}{G \cdot t} ds = 0 \tag{55}$$

The unknown shear flow, i.e., the redundant, at the cut is denoted q_o. Because the cut cross-section is open, the shear flow at other locations is:

$$q_s(s) = q_o + \frac{V}{I} \cdot Q_{det}(s) \tag{56}$$

where Q_{det} is the first moment of area with zero value at the cut, evaluated according to

$$Q = \int_0^s z \cdot dA = \int_0^s z \cdot t \cdot ds \tag{57}$$

Substitution of Eq. (56) into Eq. (55) yields

$$u = \oint \frac{q_o}{G \cdot t} ds + \oint \frac{V \cdot Q_{det}}{I \cdot G \cdot t} ds = 0 \tag{58}$$

Solving for q_o yields:

$$q_o = -\frac{V}{I} \cdot \frac{\oint \frac{Q_{det}}{G \cdot t} ds}{\oint \frac{1}{G \cdot t} ds} \tag{59}$$

By comparing Eq. (59) and Eq. (46) it becomes clear that the integral fraction in Eq. (59) represents the value of the first moment of area at the cut:

$$Q_o = -\frac{\oint \frac{Q_{det}}{G \cdot t} ds}{\oint \frac{1}{G \cdot t} ds} = -\frac{\oint \frac{Q_{det}}{t} ds}{\oint \frac{1}{t} ds} \tag{60}$$

where constant shear modulus G is assumed in the last equality. According to Eq. (60), the final Q-diagram for the cross-section can be obtained as follows:

1. Draw the Q_{det}-diagram for the statically determinate (cut) cross-section by evaluating Eq. (57)
2. Obtain the numerator in Eq. (60) by integration of the statically determinate Q-diagram divided by respective thicknesses; this integration can sometimes be cumbersome but is essentially the integral of Eq. (57)
3. Obtain the denominator in Eq. (60), which is straightforward because, for example for a rectangular closed cross-section with width b, height h, and thickness t, it is simply $b/t+b/t+h/t+h/t$
4. Obtain the final Q-diagram by adding the constant Q_o to the statically determinate one from the first step, remembering the minus-sign in Eq. (60)

Another way of evaluating Eq. (60) is to first apply integration by parts to the numerator, which essentially moves the integration from Q to $1/Gt$:

$$\begin{aligned} \oint (Q) \cdot \left(\frac{1}{Gt}\right) ds &= \oint \left(\int_0^s z \cdot t \cdot d\tilde{s} \right) \cdot \left(\frac{1}{Gt}\right) ds \\ &= \left[\left(\int_0^s z \cdot t \cdot d\tilde{s} \right) \cdot \left(\int_0^s \frac{1}{Gt} \cdot d\tilde{s} \right) \right]_o - \oint (z \cdot t) \cdot \left(\int_0^s \frac{1}{Gt} \cdot d\tilde{s} \right) ds \\ &= -\oint (z \cdot t) \cdot \left(\int_0^s \frac{1}{Gt} \cdot d\tilde{s} \right) ds \end{aligned} \tag{61}$$

where the boundary term vanishes because the first moment of area for the entire cross-section is zero when z originates at the neutral axis. Substitution into Eq. (60) yields:

$$Q_o = \frac{\oint \left(\int_0^s \frac{1}{Gt} \cdot d\tilde{s} \right) \cdot z \cdot t \cdot ds}{\oint \frac{1}{G \cdot t} ds} \tag{62}$$

If the cross-section has some open parts, i.e., flanges that stick out from the part that encloses the cell, then the following must be noted: The parts of Eq. (60) that relate to the closed integral around the cell do not pick up contributions from the protruding flanges, but Q does. In other words, the integration of $z{\cdot}t$ along s in the numerator in Eq. (62) picks up contributions from the protruding flanges. This fact is reiterated shortly. For now, notice that when the material is homogeneous, so that G is constant throughout the cross-section, the expression in Eq. (62) simplifies to:

$$Q_o = \frac{\oint \left(\int_0^s \frac{1}{t} \cdot d\tilde{s} \right) \cdot z \cdot t \cdot ds}{\oint \frac{1}{t} ds} \tag{63}$$

To reorganize this expression for practical computations it is useful to define the function

$$g(s) = \frac{\int_0^s \frac{1}{t} \cdot ds}{\oint \frac{1}{t} ds} \tag{64}$$

where the denominator is a constant and the numerator varies with s. Substitution of $g(s)$ into Eq. (63) yields the following final expression for the first moment of area at the cut:

$$Q_o = \oint g(s) \cdot z \cdot t \cdot ds \tag{65}$$

It is reiterated that $g(s)$ is unaffected by protruding open parts of the cross-section, while the integration of $z{\cdot}t{\cdot}ds$ must include contributions from those parts:

$$Q_o = \oint g(s) \cdot z \cdot t \cdot ds + \sum \left(Q_{flange\#i} \cdot g_i \right) \tag{66}$$

where Q_{flange} is the first moment of area of the flange and g_i is the value of the g-function where the flange attaches to the cell. Once Q_o is computed, the shear flow at other locations is determined as for open cross-section, relative to the location of the cut:

$$q_s(s) = \frac{V}{I} \cdot \left(Q_o + Q(s) \right) \tag{67}$$

For multi-cell cross-sections, multiple cuts are introduced to make the cross-section statically determinate. Specifically, one cut and one compatibility equation is introduced for each cell. At each cut, the value of the first moment of area, Q_o, is determined either by Eq. (60), i.e., direct integration of the statically determinate Q-diagram, or by Eq. (65), e.g., integration of the auxiliary g-function. Both approaches are viable for multi-cell cross-sections, although the use of Eq. (65) can now be somewhat more error prone. The reason is the additional complication arising from the shear flow in the separation-walls that have cells on both sides.

The use of Eq. (60) is first addressed. The step-wise procedure that was described above for single-cell cross-sections is adopted here, with an important modification; the integral of Q must include both the earlier statically determinate diagram and now also the statically indeterminate shear flow in the separation walls:

$$\oint \frac{Q}{t} ds = \oint \frac{Q_{det}(s)}{t} ds + \sum \left(h_{wall\#i} \cdot \frac{Q_{o,cell\#j}}{t_{wall\#i}} \right) \tag{68}$$

where Q_{det} is the first moment of area for the statically determinate (cut) cross-section, h_{wall} and t_{wall} are the length and thickness of a wall that separates two cells, respectively, and Q_o is the redundant in the cell on the other side of the wall compared to the cell around which the integral is conducted. In other words, the total Q-value that is integrated consists of the statically determinate values, Q_{det}, plus constants Q_o around each cell. Because the shared walls couple the redundants in neighbouring cells, a system of equations is formed, which is solved for the unknown Q_o values at the cuts.

The main challenge in the use of Eq. (60) is to integrate a Q-diagram. For single-cell cross-sections, that challenge was addressed by the introduction of the g-function and evaluation of Eq. (65). This approach is also possible for multi-cell cross-sections, but it is now easier to commit sign errors. In the application of Eq. (65) it is once again noted that $g(s)$ is unaffected by protruding flanges, while $z{\cdot}t{\cdot}ds$ must include such parts. In fact, this is how the neighbouring cells enter the continuity integral around a cell. As before, the g-function is established around each cell, varying from zero to unity according to Eq. (64). Eq. (65) is then evaluated, wall-by-wall around the cell, adding contribution from the "protruding flanges" from neighbouring cells:

$$Q_o = \oint g(s) \cdot z \cdot t \cdot ds + \sum \left(\left(Q_{flange\#j} \pm Q_{o,j} \right) \cdot g_j \right) \tag{69}$$

where Q_{flange} is the statically determinate first moment of area, Qo is the contribution from the redundant in that cell, and g is the value of the g-function where the "flange" from the neighbouring cell attaches to the cell around with the compatibility integration is taking place. In short, the integration around each cell includes the parts around the other cell(s) as if they were protruding flanges.

If the shear centre is known *a priori* then another approach is possible. Assuming the cross-section has one cell, the shear flow is determined as follows. Also in this approach, a "cut" is introduced, which yields an open cross-section. The coordinate s originates at the cut and traces the cross-section around the cell. The unknown shear flow at the cut is denoted q_o, and the shear flow at all other locations is determined relative to q_o in accordance with Eq. (46), so that

$$q_s(s) = q_o + \frac{V}{I} \cdot Q(s) \tag{70}$$

Once q_s is determined at all locations of the opened cross-section, the moment of the shear flow about the known shear centre is computed as

$$\begin{aligned} T &= \oint q_s \cdot h\,ds \\ &= \oint \left(q_s + \frac{V}{I} \cdot Q \right) \cdot h\,ds \\ &= \oint q_o \cdot h\,ds + \oint \frac{V}{I} \cdot Q \cdot h\,ds \end{aligned} \tag{71}$$

where the integrals are made around the cell, starting at s=0, and $h(s)$=distance from the shear centre to the tangent line of the contour of the cross-section at s. By definition the moment, T, about the shear centre must be zero, and solving for q_o yields

$$q_o = -\frac{V}{I} \cdot \frac{\oint Q \cdot h\, ds}{\oint h\, ds} = -\frac{V}{2 \cdot A_m \cdot I} \cdot \oint Q \cdot h\, ds \tag{72}$$

where the last equality is obtained by recognizing that the integral of h around the cross-section is twice the cell area, A_m. Having the value of q_o, the shear flow is determined at other locations with Eq. (70).

Finally, a note on shear stresses in bi-axial bending. The formulas above are derived for uni-axial bending, i.e., bending about one of the principal axes of the cross-section. The problem of bi-axial bending can be decomposed into two cases of uni-axial bending by determining the principal axes and consider bending about each axis separately. However, an alternative is to leave the principal axes unknown and rather develop stress formulas that include the product of inertia, I_{yz}, which is non-zero unless y and z are the principal axes. The expression for axial stress in Eq. (39) yields the following expression for shear stress, again considering equilibrium, as done above for open cross-sections:

$$\tau = -\frac{V_y}{t} \cdot \frac{Q_z \cdot I_y - Q_y \cdot I_{yz}}{I_y \cdot I_z - I_{yz}^2} - \frac{V_z}{t} \cdot \frac{Q_y \cdot I_z - Q_z \cdot I_{yz}}{I_y \cdot I_z - I_{yz}^2} \tag{73}$$

For reference, this expression reverts to the summation of the ordinary shear stress expressions when the y and z axes are the principal axes:

$$\tau = -\frac{V_y \cdot Q_z}{I_z \cdot t} - \frac{V_z \cdot Q_y}{I_y \cdot t} \tag{74}$$

Shear Centre

To understand the shear centre concept, think of a cantilevered beam completely fixed at one end, with a point load applied at the free end. Examples of the cross-section that may be seen at the free end is shown in Figure 8. The shear centre of a cross-section, sometimes called the centre of twist, is the point in the cross-section through which the point load, P, must act to avoid rotation of the cross-section. In other words, if the point load does not act through the shear centre then the cantilevered beam will experience torsion. The coordinates of the shear centre are denoted y_{sc} and z_{sc}, and there are several techniques to determine them. The simplest case is double-symmetric cross-sections; for these cross-sections the shear centre coincides with the centroid. Conversely, if the cross-section has *one* axis of symmetry then the shear centre is located somewhere on that symmetry axis.

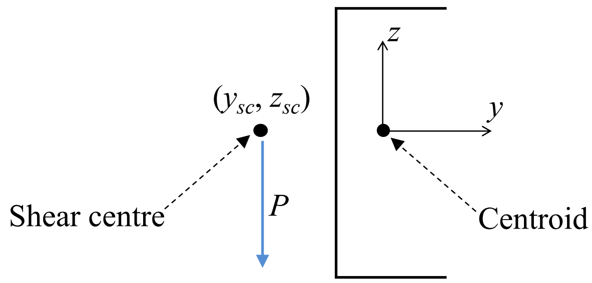

Figure 8: Shear centre.

The document on warping torsion describes one approach to determine y_{sc} and z_{sc} for thin-walled cross-sections, utilizing "omega diagrams" and the document on St. Venant torsion presents an approach for generic solid cross-section. A simpler approach is available for thin-walled cross-sections: The "shear flow" must be in equilibrium about the shear centre. Using the shear flow, the following procedure for determining the coordinates of the shear centre is suggested, provided y and z are the principal axes through the centroid of the cross-section:

1. Select an arbitrary point as trial shear centre, and let y_{sc} and z_{sc} denote the coordinates of the shear centre relative to the centroid; in other words, let y_{sc} and z_{sc} denote the distances from the centroid to the trial shear centre
2. Determine the shear flow in the cross-section due to a shear force in the z-direction, using the formulas presented above
3. Write the equation that expresses the moment of the shear flow about the trial shear centre; in general, both y_{sc} and z_{sc} will appear in this expression
4. Determine the shear flow in the cross-section due to a shear force in the y-direction, again using the formulas presented above
5. Write the equation that expresses the moment of the shear flow determined in the previous item about the trial shear centre; in general, both y_{sc} and z_{sc} will appear in this expression
6. Set the equations from Items 3 and 5 equal to zero and solve them for the two unknowns y_{sc} and z_{sc}

Only one moment equation is needed for single-symmetric cross-sections, in which case the procedure simplifies to:

1. Select an arbitrary point along the symmetry axis as trial shear centre, and let e denote the distance from the centroid to that point
2. Determine the shear flow in the cross-section due to a shear force in the direction *perpendicular* to the axis of symmetry
3. Write the equation that expresses the moment of the shear flow in Item 2 about the trial shear centre; e will appear in this expression
4. Set the equation from Items 3 equal to zero and solve for e

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