Stress-based Failure Criteria

Although the topic in this section is closely related to the theory of plasticity, which is addressed in documents, a few popular design criteria are briefly presented here. They play an important role in identifying yielding in ductile metals. As a starting point, consider the case of a uniaxial stress state, in which σ_{xx} is the only non-zero stress component. In that case, the criterion to avoid failure is

$$\sigma_{xx} < f_y \tag{1}$$

where f_y is called the yield strength. This is the same strength that appears in the more elaborate criteria below.

Tresca

Tresca postulated that yielding occurs when the maximum shear stress exceeds a material-specific threshold. In another document on this website it is shown, in the context of Mohr's circle, the maximum shear stress equals half the difference between the largest and smallest principal stress. However, instead of formulating the criterion in terms of a shear yield stress it is formulated with the regular yield stress:

$$|\sigma_3 - \sigma_1| < f_y \tag{2}$$

which implies that the shear yield stress is half the axial yield stress. This criterion is shown as a dashed line in Figure 1 for the plane stress state. Notice that the out-of-plane stress is zero. For that reason, Figure 1 avoids the use of σ_1, σ_2, and σ_3 because, in the notation in this document, σ_1 is always the largest and σ_3 is always the smallest principal stress. In two of the quadrants in Figure 1 it is the out-of-plane stress that is σ_1 or σ_3.

von Mises

This yield criterion inherits the notion of a shear-stress-based yield criterion from Tresca. However, it takes as a starting point a reformulation of the stress tensor, split into a volumetric part and a deviatoric part:

$$\sigma_{ij} = s_{ij} + p \cdot \delta_{ij} = \underbrace{\begin{bmatrix} (\sigma_{xx} - p) & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & (\sigma_{yy} - p) & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & (\sigma_{zz} - p) \end{bmatrix}}_{\text{Deviatoric stress, } \mathbf{s}} + \underbrace{\begin{bmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{bmatrix}}_{\text{Volumetric stress}} \tag{3}$$

where

$$p = \frac{\sigma_{ii}}{3} = \frac{\sigma_{xx} + \sigma_{yy} + \sigma_{zz}}{3} \tag{4}$$

The deviatoric stress tensor represents distortion, while the volumetric part represents pressure. The derivation of von Mises' yield criterion proceeds to carry out an eigenvalue analysis for the deviatoric stress tensor in the same way as the principal stresses were obtained for the ordinary stress tensor. However, instead of the notation I_i the deviatoric stress invariants are denoted J_i:

$$\left|s_{ij} - \lambda \cdot \delta_{ij}\right| = \lambda^3 - J_1 \cdot \lambda^2 - J_2 \cdot \lambda - J_3 = 0 \tag{5}$$

where vertical bars identify the determinant and the deviatoric stress invariants are

$$J_1 = \sigma_{xx} + \sigma_{yy} + \sigma_{zz} \tag{6}$$

$$J_2 = \frac{1}{6} \cdot \left(\sigma_{xx} - \sigma_{yy}\right)^2 + \frac{1}{6} \cdot \left(\sigma_{yy} - \sigma_{zz}\right)^2 + \frac{1}{6} \cdot \left(\sigma_{zz} - \sigma_{xx}\right)^2 + \sigma_{xy}^2 + \sigma_{yz}^2 + \sigma_{zx}^2 \tag{7}$$

$$I_3 = |\mathbf{s}| \tag{8}$$

The von Mises criterion defines yielding in terms of the second deviatoric stress invariant. This is the basis for label "J2 plasticity" used in other documents on this website. It is postulated that yielding occurs when J_2 exceeds a material constant. To synchronize with the yield stress, f_y, for the uniaxial stress state, this constant is selected so that yielding is assumed to occur when

$$J_2 = \frac{f_y^2}{3} \tag{9}$$

Rearranged, the von Mises yield criterion is

$$\sqrt{3 \cdot J_2} < f_y \tag{10}$$

For the plane stress state,

$$J_2 = \frac{\sigma_{xx}^2}{3} + \frac{\sigma_{yy}^2}{3} - \frac{\sigma_{xx}\sigma_{yy}}{3} + \sigma_{xy}^2 \tag{11}$$

Substitution of Eq. (11) into Eq. (10) provides the von Mises yield criteron for plane stress conditions:

$$\sqrt{\sigma_{xx}^2 + \sigma_{yy}^2 - \sigma_{xx}\sigma_{yy} + 3 \cdot \sigma_{xy}^2} < f_y \tag{12}$$

This criterion is visualized by a solid line in Figure 1 along with the Tresca criterion. For the uniaxial case, J_2 is given in Eq. (9) and the von Mises criterion boils down to $|\sigma| < f_y$.

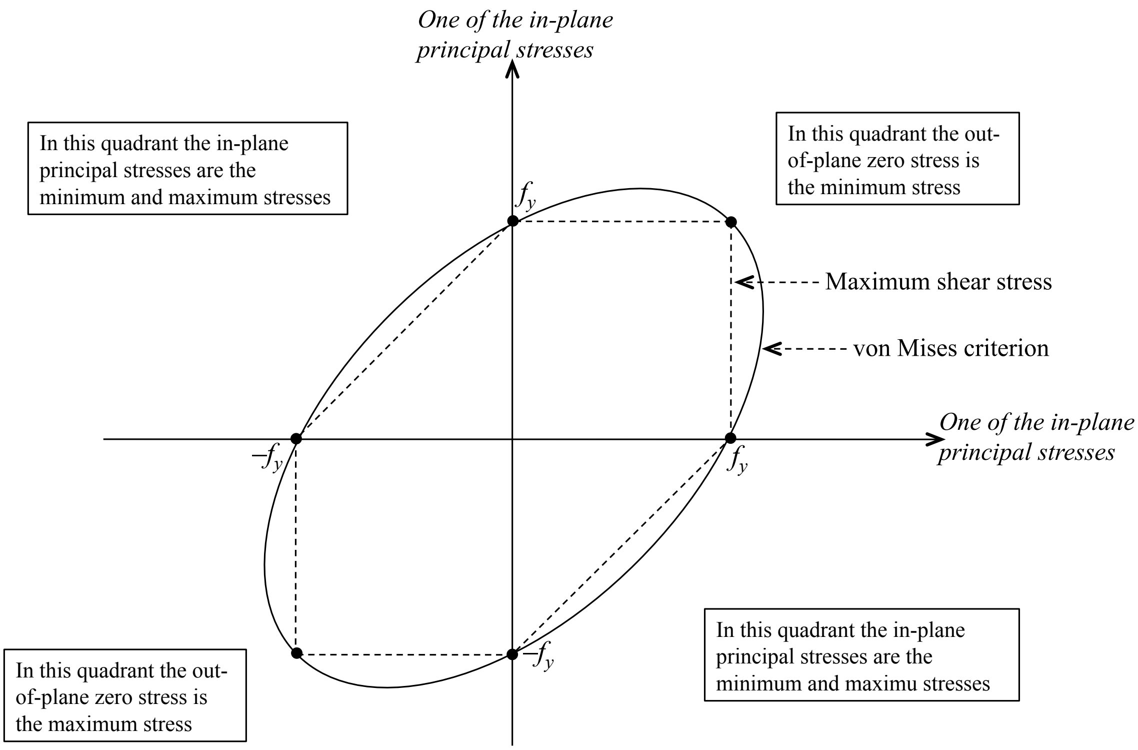

Figure 1: Tresca and von Mises yield criteria for 2D stress state.

Drucker-Prager

This failure criterion is often employed in geotechnical engineering and extends the von Mises criterion to include the volumetric stress, in addition to the deviatoric stress. It is written in terms of the second deviatoric stress invariant, J_2, and the first regular stress invariant, I_1. Just like the von Mises criterion can be written as an equality as

$$\sqrt{3 \cdot J_2} = f_y \tag{13}$$

the Drucker-Prager the criterion is written

$$\sqrt{J_2} = a + b \cdot I_1 \tag{14}$$

where a and b are material constants. J_2 is defined earlier in this document and from the document on stress transformations,

$$I_1 = \sigma_{xx} + \sigma_{yy} + \sigma_{zz} \tag{15}$$

which means that the Drucker-Prager criterion reads, in the uniaxial case,

$$\frac{1}{\sqrt{3}} \cdot |\sigma| = a + b \cdot \sigma \tag{16}$$

Given the presence of two material constants, that borderline between safe and fail can be expressed on the tension side (positive stress) as

$$\frac{1}{\sqrt{3}} \cdot f_{y\ \text{(tension)}} = a + b \cdot f_{y\ \text{(tension)}} \tag{17}$$

and on the compression side (negative stress) as

$$\frac{1}{\sqrt{3}} \cdot f_{y\ \text{(compression)}} = a - b \cdot f_{y\ \text{(compression)}} \tag{18}$$

Solving Eqs. (17) and (18) for *a* and *b* yields

$$a = \frac{2}{\sqrt{3}} \cdot \frac{f_{y\ \text{(tension)}} \cdot f_{y\ \text{(compression)}}}{\left(f_{y\ \text{(tension)}} + f_{y\ \text{(compression)}}\right)} \tag{19}$$

and

$$b = \frac{1}{\sqrt{3}} \cdot \frac{f_{y\ \text{(tension)}} - f_{y\ \text{(compression)}}}{\left(f_{y\ \text{(tension)}} + f_{y\ \text{(compression)}}\right)} \tag{20}$$

If the yield stress is the same on the tension side and the compression side then

$$a = \frac{f_y}{\sqrt{3}} \tag{21}$$

and

$$b = 0 \tag{22}$$

so that the criterion reverts to the von Mises criterion

$$|\sigma| = f_y \tag{23}$$