Bouc-Wen Material Model

This material model is characterized by a smooth transition from elastic to yielding material behaviour. It is also characterized by a set of material parameters that may lead to unphysical response, unless proper values are provided, so caution is required. The model was developed by Bouc (1971) and Wen (1976). Baber & Noori (1985) presented a version of the model that accommodates strength and stiffness degradation. I implemented that model in OpenSees, but the version that Andreas Schellenberg later implemented in OpenSees is a neat alternative formulated in terms of common stiffness and strength parameters, and is adopted in this document. In the Bouc-Wen model, the stress is written as the following sum of a linear part and a hysteretic part:

$$\sigma = \alpha \cdot E \cdot \varepsilon + (1-\alpha) \cdot f_y \cdot z \tag{1}$$

where α=fraction E that serves as second-slope stiffness, E=modulus of elasticity, ε=strain, f_y=yield stress, and z=hysteresis evolution variable. Under assumptions that are adopted shortly, z takes on values between –1 and 1. Figure 1 shows the implication of Eq. (1) in a stress-strain diagram, where the green line is the stress for a given strain. The figure illustrates the smooth transitions, and the asymptotes to which the stress converges for different ε and z values.

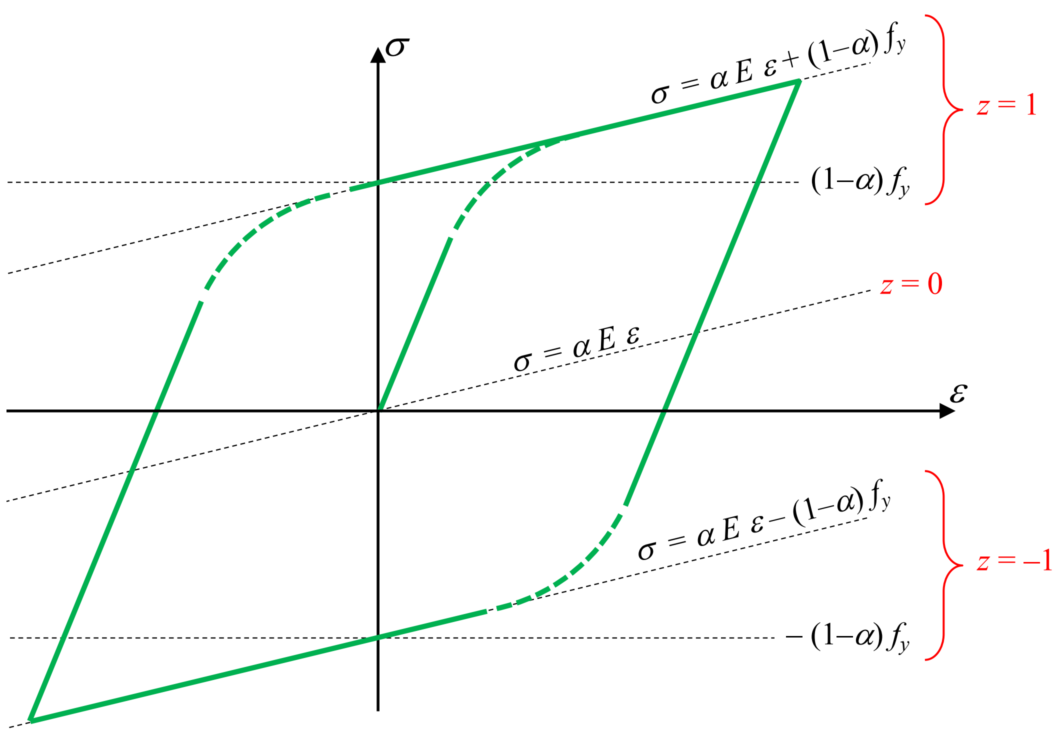

Figure 1: Illustration of the terms in the stress expression.

The hysteresis evolution variable is governed by the differential equation

$$\dot{z} = \frac{1}{\varepsilon_y} \cdot (\dot{\varepsilon} - \gamma \cdot \dot{\varepsilon} \cdot |z|^\eta - \beta \cdot |\dot{\varepsilon}| \cdot z^\eta) \tag{2}$$

where $\varepsilon_y = f_y/E$=yield strain, and γ, β, and η are material parameters. In this context, ε_y is essentially a scaling factor that provides a transition from elastic to yielding within appropriate strain values. The specification of γ and β in a manner such that the sum of their individual absolute values equal unity implies that z takes on values between –1 and 1. That is because Eq. (2) says the rate of change of z approaches zero as z approaches –1 or 1. The value $\gamma=\beta=0.5$ is appropriate for a bilinear material with smooth transitions. Note that η is a positive integer; it governs the sharpness of the transitions shown by dashed lines in Figure 1. The higher value of η, the sharper the transition from elastic to yielding. As will be seen shortly, it is helpful to isolate the strain rate in Eq. (2), which yields the following revised version of Eq. (2):

$$\dot{z} = \frac{1}{\varepsilon_y} \cdot \left(1 - \left(\gamma + \beta \cdot \text{sign}(\dot{\varepsilon} \cdot z)\right) \cdot |z|^\eta\right) \cdot \dot{\varepsilon} \tag{3}$$

The signum function appears in order to accommodate the switch of the absolute value operator in the last term on the right-hand side of Eq. (2). The rewrite in Eq. (3) is helpful because $\dot{\varepsilon} = d\varepsilon/dt = \Delta\varepsilon/\Delta t$, and the strain increment, $\Delta\varepsilon$, is always available to all material models in nonlinear analysis. Eq. (3), which governs the evolution of z, is now discretized using the Backward Euler scheme. This is done in order to obtain z_{n+1} from z_n at increment n. For a generic differential equation $\dot{z} = r(z(t))$ the algorithm reads $z_{n+1}=z_n+\Delta t \cdot r(z_{n+1})$. Given the right-hand-side of Eq. (3), and the fact that $\dot{\varepsilon} = \Delta\varepsilon/\Delta t$, the Backward Euler scheme reads, after recognizing that Δt cancels:

$$z_{n+1} = z_n + \frac{1}{\varepsilon_y} \cdot \left(1 - \left(\gamma + \beta \cdot \text{sign}(\dot{\varepsilon}_{n+1} \cdot z_n)\right) \cdot |z_{n+1}|^\eta\right) \cdot \Delta\varepsilon_{n+1} \tag{4}$$

As expected for the Backward Euler scheme, z_{n+1} appears on both sides of Eq. (4). For a moment, let z_{n+1} be labelled x, and consider the Newton algorithm $x_{j+1}=x_j-f(x_j)/f'(x_j)$ for the generic root-finding problem $f(x)=0$, where f' means df/dx. The function f is, from Eq. (4)

$$f = z_{n+1} - z_n - \left(1 - \left(\gamma + \beta \cdot \text{sign}(\Delta\varepsilon_{n+1} \cdot z_n)\right) \cdot |z_{n+1}|^\eta\right) \cdot \frac{\Delta\varepsilon_{n+1}}{\varepsilon_y} \tag{5}$$

where $\Delta\varepsilon_{n+1}$ takes the place of $\dot{\varepsilon}_{n+1}$ in the signum function because the sign is determined by $\Delta\varepsilon_{n+1}$. The derivative of f is

$$\frac{df}{dz_{n+1}} = 1 + \eta \cdot \text{sign}(z_{n+1}) \cdot \left(\gamma + \beta \cdot \text{sign}(\Delta\varepsilon_{n+1} \cdot z_n)\right) \cdot |z_{n+1}|^{\eta-1} \frac{\Delta\varepsilon_{n+1}}{\varepsilon_y} \tag{6}$$

Once z_{n+1} is determined, the stress is obtained from Eq. (1).

Tangent

In addition to the stress, the material algorithm must return the tangent stiffness. The tangent stiffness is used in the global Newton-Raphson scheme to compute the nonlinear structural response. The accuracy and consistency of the tangent stiffness is vital for the convergence rate of the Newton-Rapshon algorithm; however, it is also crucial for sensitivity analysis with the direct differentiation method. The starting point for the derivation of the tangent for the Bouc-Wen material model is the derivative of Eq. (1) with respect to the current strain:

$$\frac{d\sigma_{n+1}}{d\varepsilon_{n+1}} = \alpha \cdot E + (1-\alpha) \cdot f_y \cdot \frac{dz_{n+1}}{d\varepsilon_{n+1}} \tag{7}$$

When calculating the last factor, i.e., $\partial z_{n+1}/\partial \varepsilon_{n+1}$, the concept of a continuum tangent must be carefully distinguished from the algorithmically consistent tangent. It is the latter that should be implemented on the computer. For reference, the continuum tangent is first presented here, although it would be a mistake to employ it in the Newton-Raphson algorithm. Doing so would slow convergence and yield inaccurate response sensitivities from the direction differentiation method. To understand what the continuum tangent is, consider the decomposition of $\dot{z}$ using the chain rule of differentiation in the following manner:

$$\dot{z} = \frac{dz}{dt} = \frac{dz}{d\varepsilon}\frac{d\varepsilon}{dt} = \frac{dz}{d\varepsilon}\dot{\varepsilon} \tag{8}$$

Comparing Eq. (8) with Eq. (3) shows that

$$\frac{dz}{d\varepsilon} = \frac{1}{\varepsilon_y} \cdot \left(1 - \left(\gamma + \beta \cdot \text{sign}(\dot{\varepsilon} \cdot z)\right) \cdot |z|^{\eta}\right) \tag{9}$$

Substituting Eq. (9) into Eq. (7) yields the continuum tangent. Conversely, the algorithmically consistent tangent is obtained by differentiating the previously presented equations that compute z_{n+1} with respect to the strain, and substituting the result, i.e., $\partial z_{n+1}/\partial \varepsilon_{n+1}$, into Eq. (7). Two approaches, effectively leading to the same tangent value, are explored. First, recognize that the Newton-Raphson algorithm within the material calculates z_{n+1} using the iterative algorithm

$$(z_{n+1})_{j+1} = (z_{n+1})_j - \frac{f\left((z_{n+1})_j\right)}{\dfrac{df\left((z_{n+1})_j\right)}{dz_{n+1}}} \tag{10}$$

Any change in z_{n+1} after convergence would emanate solely from a change in the fraction in the last term of Eq. (10). That means the sought derivative is the derivative of that fraction with respect to the strain, i.e.,

$$\frac{dz_{n+1}}{d\varepsilon_{n+1}} = -\frac{d}{d\varepsilon_{n+1}}\left(\frac{f(z_{n+1})}{\dfrac{df(z_{n+1})}{dz_{n+1}}}\right) \tag{11}$$

By adopting the notation $f = f(z_{n+1})$ and $f' = df(z_{n+1})/dz_{n+1}$ the chain rule of differentiation applied to Eq. (11) yields

$$\frac{dz_{n+1}}{d\varepsilon_{n+1}} = -\left(\frac{\left(\frac{df}{d\varepsilon_{n+1}}\right)}{f'} - \frac{f}{(f')^2}\cdot\left(\frac{df'}{d\varepsilon_{n+1}}\right)\right) \tag{12}$$

where

$$\frac{df}{d\varepsilon_{n+1}} = -\big(1-\big(\gamma+\beta\cdot\mathrm{sign}(\Delta\varepsilon_{n+1}\cdot z_n)\big)\cdot|z_{n+1}|^{\eta}\big)\cdot\frac{1}{\varepsilon_y} \tag{13}$$

and

$$\frac{df'}{d\varepsilon_{n+1}} = \eta\cdot\mathrm{sign}(z_{n+1})\cdot\big(\gamma+\beta\cdot\mathrm{sign}(\Delta\varepsilon_{n+1}\cdot z_n)\big)\cdot|z_{n+1}|^{\eta-1}\frac{1}{\varepsilon_y} \tag{14}$$

Eq. (12) substituted into Eq. (7) is the sought algorithmically consistent tangent. Another approach is to employ Eq. (5) in conjunction with the condition f=0 to write

$$z_{n+1} = z_n + \big(1-\big(\gamma+\beta\cdot\mathrm{sign}(\Delta\varepsilon_{n+1}\cdot z_n)\big)\cdot|z_{n+1}|^{\eta}\big)\cdot\frac{\varepsilon_{n+1}-\varepsilon_n}{\varepsilon_y} \tag{15}$$

Differentiation of Eq. (15) with respect to the current strain yields

$$\begin{aligned}\frac{dz_{n+1}}{d\varepsilon_{n+1}} &= \big(\gamma+\beta\cdot\mathrm{sign}(\Delta\varepsilon_{n+1}\cdot z_n)\big)\cdot\eta\cdot|z_{n+1}|^{\eta-1}\cdot\mathrm{sign}(z_{n+1})\cdot\frac{dz_{n+1}}{d\varepsilon_{n+1}}\cdot\frac{\varepsilon_{n+1}-\varepsilon_n}{\varepsilon_y}\\ &\quad+\big(1-\big(\gamma+\beta\cdot\mathrm{sign}(\Delta\varepsilon_{n+1}\cdot z_n)\big)\cdot|z_{n+1}|^{\eta}\big)\cdot\frac{1}{\varepsilon_y}\end{aligned} \tag{16}$$

Solving Eq. (16) for $\partial z_{n+1}/\partial\varepsilon_{n+1}$ yields a value identical to that obtained from Eq. (11).

References

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