Stiffness Method

In modern structural analysis the stiffness method is vital because it is the basis for computational analysis of structures. Its extension to problems beyond trusses and frames is called the "finite element method," which is at the pinnacle of structural analysis software. In the same way as the flexibility method is the quintessential force method, the stiffness method is the quintessential displacement method. While the stiffness method has its biggest strength when implemented on the computer, it can also be employed in hand calculations. That is the focus of this document.

In the stiffness method, equilibrium equations are established and solved for unknown joint displacements and rotations. The joints are often referred to as nodes, and the unknowns are called degrees of freedom (DOFs). The stiffness method is popular because no subjective choices are made when determining the DOFs. That is different with the selection of redundants in the flexibility method. In computer implementations of the stiffness method, the computer straightforwardly assigns a pre-defined number of DOFs to each node and establishes a linear system of equilibrium equations automatically. In hand calculations we often neglect axial deformations. That is harder for the computer to do, but it reduces the number of unknowns in hand calculations without significantly affecting the accuracy of the results.

The key steps of the stiffness method are:

1. Determine the DOFs of the structure, i.e., the unknown displacements and rotations, which are collected in the vector **u**
2. Establish the stiffness matrix, **K**, which contains the stiffness coefficients explained below
3. Establish the load vector, **F**, which contains the applied loads
4. Solve the system of equilibrium equations, **Ku**=**F**, to obtain the unknown displacements and rotations
5. Determine the element end forces from **u** and draw the section force diagrams

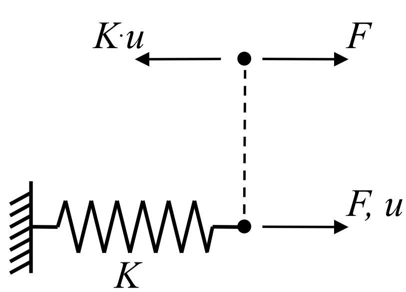

Figure 1: A spring is a one-DOF problem.

Equilibrium Equations

As an introduction to the stiffness method, first consider a simple problem with only one DOF. For that purpose, the problem of a spring with stiffness K shown in Figure 1 is addressed. Let F and u be the force and displacement along the DOF, respectively. As the spring is being pulled or compressed, a force equal to $K \cdot u$ develops in the spring. This force in the spring balances the applied force. Consequently, the equilibrium equation for this problem is

$$K \cdot u = F \tag{1}$$

where the stiffness coefficient, K, represents the force due to a unit displacement. Also observe that applied loads enter the right-hand side of Eq. (1) as positive when they act in the direction of the DOF. In other words, the applied load, F, in Figure 1 is positive because it acts in the direction of the DOF.

Next, consider a 2-DOF problem, i.e., a problem with two DOFs, numbered 1 and 2. This means that two equilibrium equations are needed. The following two generic equilibrium equations are established by requiring equilibrium along the two DOFs:

$$\begin{aligned} K_{11}\, u_1 + K_{12}\, u_2 &= F_1 \\ K_{21}\, u_1 + K_{22}\, u_2 &= F_2 \end{aligned} \tag{2}$$

which can be written $K_{ij}\ u_j = F_i$, where K_{ij} = force along DOF number i due to a unit displacement or rotation along DOF number j, u_j = unknown displacement or rotation along DOF number j, and F_i = force along DOF number i due to external loads. The system of equilibrium equations in Eq. (2) is written equivalently in matrix notation as

$$\mathbf{Ku}=\mathbf{F} \tag{3}$$

where **K**=stiffness matrix, **u**=displacement vector, which contains the unknown displacements and rotations along the DOFs, and **F**=load vector.

Establishing the Stiffness Matrix

Once the DOFs of the structure are identified, in accordance with the document on degrees of indeterminacy, the stiffness matrix is established as follows:

1. Sketch the displaced shape of the structure for a unit displacement or rotation along DOF number j, with all other DOFs clamped
2. Determine the force along every DOF to maintain this displaced shape, i.e., K_{ij}, which form column number j of the stiffness matrix
3. Carry out Step 1 and 2 for all DOFs to establish all columns of the stiffness matrix
4. Check that the final stiffness matrix is symmetric and that it has positive components on the diagonal

The key challenge is Step 2, i.e., the determination of forces along the DOFs to maintain the displaced shape. For each DOF it is necessary to account for every force, including moments, from every member. Figure 2 provides stiffness-values for a few fundamental beam cases to assist this process. The stiffness values are derived from solving the differential equation, or equivalently by employing the slope-deflection equation, or the principle of virtual work. Each quantity in the auxiliary beam case is multiplied by the imposed displacement Δ or rotation θ to obtain the actual value of the force or moment. Axial stiffness is omitted from Figure 2 because in the classical stiffness method it is straightforward and often sufficiently accurate to neglect axial deformations in frame members. Axial deformations are often orders of magnitude less than the bending (flexural) deformations because the axial stiffness is large compared to the bending stiffness for many frame members. Axial deformations are neglected simply by considering the members infinitely stiff in the axial direction, so that the associated DOFs of the structure are removed.

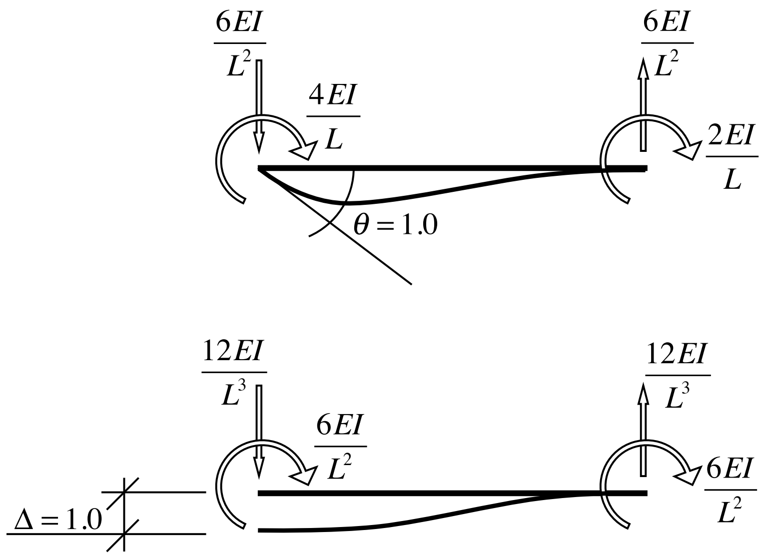

Figure 2: Stiffness values for fundamental beam cases.

Establishing the Load Vector

The equilibrium equations **Ku**=**F** are conceptually and pedagogically appealing. However, an additional discussion is warranted on the inclusion of applied loads. Two cases must be considered:

A. Point loads acting directly along a DOF
B. Distributed loads or point loads acting somewhere along a structural member

Case A is straightforwardly addressed by inserting the point load into the appropriate position of the load vector. If the load acts along the DOF, then it is positive. When addressing Case B, it is helpful to think of the stiffness method as a three-step process:

1. Clamp all DOFs
2. Establish equilibrium equations, including contributions from loads, for the clamped structure
3. Release all DOFs at once, i.e., solve **Ku**=**F** for **u**.

The fact that the equilibrium equations are established for the clamped structure means it is useful to think of "clamping forces" along the DOFs, caused by the applied loads. Example cases are provided in Figure 3. In these notes, the clamping forces are identified by a **F**-bar symbol, leading to the following expanded equilibrium equations:

$$\mathbf{Ku} + \bar{\mathbf{F}} = \grave{\mathbf{F}} \tag{4}$$

This means that $\grave{\mathbf{F}}$ contains point loads acting along the DOFs (Case A) while $\bar{\mathbf{F}}$ contains clamping forces from distributed loads, or point loads, acting somewhere along structural members (Case B). If we wish to write the equilibrium equations with all external loads on the right-hand side then Eq. (4) is reorganized to read

$$\mathbf{Ku} = \underbrace{\grave{\mathbf{F}} - \bar{\mathbf{F}}}_{\substack{\text{Total}\\ \text{load}\\ \text{vector}}} = \mathbf{F} \tag{5}$$

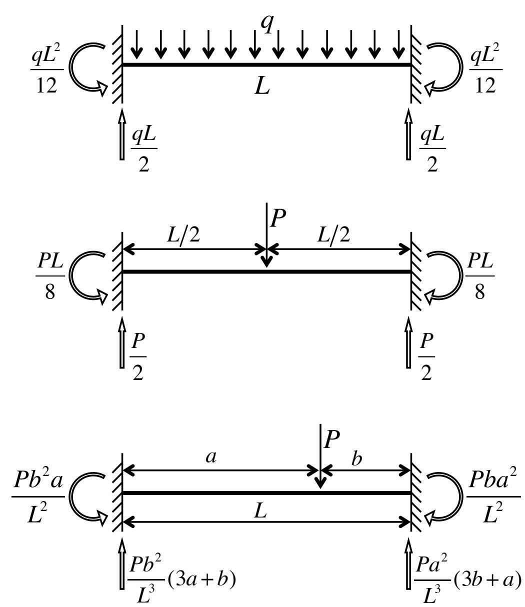

Figure 3: Clamping forces for a few beam cases.

Determining Member Forces

Upon establishing the stiffness matrix, **K**, and the load vector, **F**, the linear system of equilibrium equations **Ku**=**F** is solved to obtain **u**. The ultimate goal is to determine the internal forces in each member in order to draw the bending moment diagram, etc. One way to do that is to utilize the slope-deflection equation, derived in another document on this website:

$$M_{NF} = \frac{2EI}{L} \cdot \left(2\theta_N + \theta_F - 3 \cdot \psi\right) + FEM_{NF} \tag{6}$$

For each member, the applicable nodal displacements and rotations from **u** are inserted as θ_N, θ_F, and ψ in Eq. (6).