Slope Deflection Method

This method serves two purposes. On one hand, it is method for analyzing statically indeterminate structures. Equally important, it provides an introduction to other "displacement-based" methods. In fact, it is a good introduction to the stiffness method, and its extension called the finite element method, which is another displacement-based method that is vital in modern structural analysis. Like the stiffness method, the slope-deflection method formulates equilibrium equations along degrees of freedom (DOFs). The method consists of five steps:

1. Identify the DOFs, i.e., the unknown displacements & rotations
2. Establish equilibrium equations manually, along each DOF
3. Insert slope-deflection equations into the equilibrium equations
4. Solve for the unknown displacements & rotations
5. Substitute that solution into slope-deflection equations to get end moments

For frames with only rotational DOFs, the execution of that procedure is straightforward, as is shown in examples in class. For frames with "sidesway," i.e., displacement DOFs, we need to establish "shear equilibrium" equations in addition to moment equilibrium equations.

Derivation of the Slope-Deflection Equation by the Unit Virtual Load Method

The slope-deflection equation exposes the relationship between end-rotation and end-moment for a frame member. For this purpose, consider a horizontal beam from *A* to *B* of length *L* subjected to a bending moment at *A* called M_{AB} and a bending moment at *B* called M_{BA}. Furthermore, let clockwise end-moments be positive. To derive the slope-deflection equation by virtual work, start by applying a unit virtual moment at A, which reveals the following rotation:

$$\theta_A = \frac{1}{3EI} \cdot 1 \cdot M_{AB} \cdot L - \frac{1}{6EI} \cdot 1 \cdot M_{BA} \cdot L \tag{1}$$

Next, apply a unit virtual load at B, which reveals the following rotation there:

$$\theta_B = \frac{1}{3EI} \cdot 1 \cdot M_{BA} \cdot L - \frac{1}{6EI} \cdot 1 \cdot M_{AB} \cdot L \tag{2}$$

Eqs. (1) and (2) expresses end-rotations in terms of end-moments. To obtain the slope-deflection equations, Eqs. (1) and (2) are solved to express end-moments in terms of end-rotations:

$$M_{AB} = \frac{4EI}{L}\theta_A + \frac{2EI}{L}\theta_B = \frac{2EI}{L} \cdot \left(2\theta_A + \theta_B\right) \tag{3}$$

$$M_{BA} = \frac{4EI}{L}\theta_B + \frac{2EI}{L}\theta_A = \frac{2EI}{L} \cdot \left(2\theta_B + \theta_A\right) \tag{4}$$

It is observed that the end-moment at A has twice the contribution from the rotation at A compared with the rotation at B. More generally, the contribution from a rotation at the "near" end is twice that of the rotation at the "far" end. By introducing the letters N and F for near and far, respectively, the general slope-deflection equation reads

$$M_{NF}=\frac{2EI}{L}\cdot\left(2\theta_N+\theta_F\right) \tag{5}$$

If the member is subjected to distributed loads then the end-moment is amended:

$$M_{NF}=\frac{2EI}{L}\cdot\left(2\theta_N+\theta_F\right)+FEM_{NF} \tag{6}$$

where FEM_{NF} is the "fixed-end moment," i.e., the end-moment at the near end when $\theta_N=\theta_F=0$, that is, for a fixed-fixed beam. A table of fixed-end moments is provided in one of the auxiliary documents on this website. In terms of member deformation, the slope-deflection equation in Eq. (6) includes only end-rotations. End displacements are introduced in the form of "chord rotation." The chord is the straight line that is drawn between the member ends. Observe that when the chord rotates clockwise while the ends are fixed against rotation, the member actually undergoes bending. In particular, the member experiences locally negative end rotations. By denoting the chord rotation by ψ, the end rotations due to the chord rotation are:

$$\theta_A=\theta_B=-\psi \tag{7}$$

Substitution of Eq. (7) into the slope-deflection equation in Eq. (6) yields

$$M_{NF}=\frac{2EI}{L}\cdot\left(2\theta_N+\theta_F-3\cdot\psi\right)+FEM_{NF} \tag{8}$$

which is the final form of the slope-deflection equation, in which θ_N and θ_F must be interpreted as rotations relative to the global coordinate system, not relative to the chord.

Derivation of the Slope-Deflection Equation by the Moment-Area Method

Consider a horizontal beam from A to B of length L subjected to some distributed load q. Let the bending moment at A be called M_{AB} and the bending moment at B be called M_{BA}. Furthermore, let clockwise end-moments be positive. To start deriving the slope-deflection equation note that the following relationships between the end moments and the tangential deviations at the ends:

$$\theta_A=\frac{t_{BA}}{L} \quad \text{and} \quad \theta_B=\frac{t_{AB}}{L} \tag{9}$$

The moment-area method provides:

$$t_{BA}=\underbrace{\left(\frac{M_{AB}}{EI}\frac{L}{2}\right)}_{\text{"area"}}\cdot\underbrace{\left(\frac{2L}{3}\right)}_{\text{"arm"}}-\left(\frac{M_{BA}}{EI}\frac{L}{2}\right)\cdot\left(\frac{L}{3}\right)+\frac{A_M\cdot x_B}{EI} \tag{10}$$

where the first term is due to the moment acting at A, while the second term is due to the moment acting at B. The third term is the contribution from the bending moment diagram that is solely due to the distributed load, i.e., as if it were acting on a simply supported beam. A_M is the area of the latter bending moment diagram, and x_B is its distance from its centroid to B. Similarly:

$$t_{AB} = \left(\frac{M_{BA}}{EI} \frac{L}{2} \right) \cdot \left(\frac{2L}{3} \right) - \left(\frac{M_{AB}}{EI} \frac{L}{2} \right) \cdot \left(\frac{L}{3} \right) - \frac{A_M \cdot x_A}{EI} \tag{11}$$

Substitution of Eq. (9) into Eqs. (10) and (11), and solving for M_{AB} and M_{BA}, yields:

$$M_{AB} = \frac{2EI}{L} \left(2\theta_A + \theta_B \right) + \frac{2 \cdot A_M \cdot x_A}{L^2} - \frac{4 \cdot A_M \cdot x_B}{L^2} \tag{12}$$

$$M_{BA} = \frac{2EI}{L} \left(2\theta_B + \theta_A \right) + \frac{4 \cdot A_M \cdot x_A}{L^2} - \frac{2 \cdot A_M \cdot x_B}{L^2} \tag{13}$$

In short-hand notation, these equations are summarized in the final slope-deflection equation:

$$M_{NF} = \frac{2EI}{L} \cdot \left(2 \cdot \theta_N + \theta_F \right) + FEM_{NF} \tag{14}$$

where the subscript N is read "near end," the subscript F is read "far end," and FEM is an abbreviation for "fixed-end moment." An auxiliary document in the structural analysis notes contain a table of fixed-end moment, which are computed by the right-most two terms in Eqs. (12) and (13). For example, for a beam with a point-load at midspan:

$$A_M \cdot x_A = A_M \cdot x_B = \left(\frac{PL}{4} \cdot L \cdot \frac{1}{2} \right) \cdot \left(\frac{L}{2} \right) = \frac{PL^3}{16} \quad \Rightarrow \quad FEM_{AB} = -\frac{PL}{8} \tag{15}$$

For frames with unknown joint displacements, called frames with sidesway, the slope-deflection equation is modified to account for joint displacements. In particular, the rotations in Eq. (9) are amended with the chord rotation:

$$\theta_A = \underbrace{\frac{t_{BA}}{L}}_{\substack{\text{deviation}\\ \text{from}\\ \text{chord}}} + \underbrace{\psi_{AB}}_{\substack{\text{chord}\\ \text{rotation}}} \quad \text{and} \quad \theta_B = \underbrace{\frac{t_{AB}}{L}}_{\substack{\text{deviation}\\ \text{from}\\ \text{chord}}} + \underbrace{\psi_{AB}}_{\substack{\text{chord}\\ \text{rotation}}} \tag{16}$$

where the chord rotation is positive when it is clockwise. This leads to a new term in the final form of the slope-deflection equation:

$$M_{NF} = \frac{2EI}{L} \cdot \left(2 \cdot \theta_N + \theta_F - 3 \cdot \psi_{AB} \right) + FEM_{NF} \tag{17}$$

Interpretation of the Slope-Deflection Equation

Several insights are gained from the slope-deflection equation, and these insights reappear frequently in the ubiquitous stiffness method. First, it is observed that the moment-rotation relationship for a fixed-pinned beam that is rotated at the pinned end is

$$M = \frac{4EI}{L} \tag{18}$$

It is also noticed that the moment at the other end, i.e., the end that is held fixed, is

$$M = \frac{2EI}{L} \tag{19}$$

This leads to the general observation that a moment applied at one end carries over to half that value at the other end:

$$COM = \frac{1}{2}M \tag{20}$$

where *COM* is an abbreviation for carry-over moment.