Timoshenko Beam Elements

Timoshenko beam theory amends Euler-Bernoulli theory to include shear deformations. The objective in this document is to establish the stiffness matrix for beam bending that includes shear deformation. The element is shown in Figure 1 in its basic and local configurations. Two approaches are possible, both addressed in this document:

- Apply the principle of virtual forces:

$$\delta W_{ext} = \delta W_{int} \quad \Rightarrow \quad \delta F \cdot \Delta = \int_0^L \delta M \cdot \frac{M}{EI} \mathrm{d}x + \int_0^L \delta V \cdot \frac{V}{GA_v} \, \mathrm{d}x \tag{1}$$

- Apply the principle of virtual displacements:

$$\delta W_{int} = \delta W_{ext} \quad \Rightarrow \quad \int_V \sigma \, \delta\varepsilon \, dV + \int_V \tau \, \delta\gamma \, dV = \int_0^L q_z \, \delta w \, dx \tag{2}$$

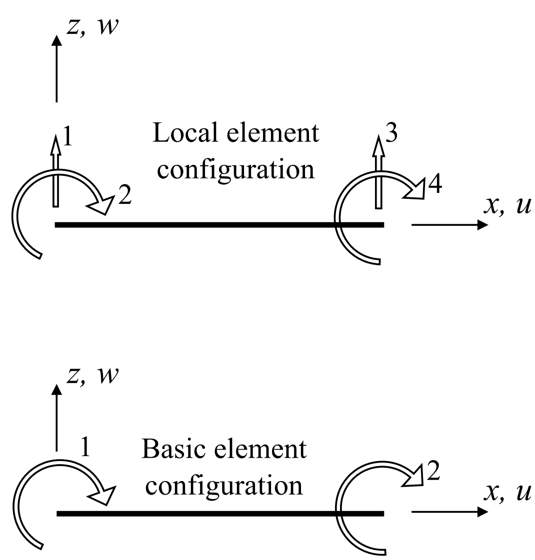

Figure 1: Beam element configurations for 2D structural analysis.

Principle of Virtual Forces

To employ this principle, it is necessary to first establish the flexibility matrix, followed by inversion to obtain the stiffness matrix. This takes place in the basic element configuration. As a starting point, recall that the stiffness coefficient k_{ij} is "force along degree of freedom number i due to a unit displacement/rotation along degree of freedom number j." However, the unit virtual load method yields deformations, not forces. For that reason it is convenient to first establish the flexibility coefficients f_{ij} instead of stiffness coefficients k_{ij}. Subsequently, the flexibility matrix is inverted to obtain the stiffness matrix. Figure 2 shows the virtual and real section force diagrams used to obtain f_{11}, f_{12}, f_{21}, and f_{22}, where f_{ij} is the displacement or rotation along degree of freedom number i due to a unit force along degree of freedom number j. Each flexibility coefficient is computed by evaluating the virtual work integral

$$f_{ij} = \int_0^L \frac{M \cdot \delta M}{EI}\,\mathrm{d}x + \int_0^L \frac{V \cdot \delta V}{GA_v}\,\mathrm{d}x \tag{3}$$

By employing the "quick integration" formulas from the unit virtual work method, the following results are obtained:

$$f_{11} = \frac{1}{3EI}\cdot 1 \cdot 1 \cdot L + \frac{1}{GA_v}\cdot\left(-\frac{1}{L}\right)\cdot\left(-\frac{1}{L}\right)\cdot L = \frac{L}{3EI} + \frac{1}{GA_v L} \tag{4}$$

$$f_{21} = -\frac{1}{6EI}\cdot 1 \cdot 1 \cdot L + \frac{1}{GA_v}\cdot\left(-\frac{1}{L}\right)\cdot\left(-\frac{1}{L}\right)\cdot L = -\frac{L}{6EI} + \frac{1}{GA_v L} \tag{5}$$

$$f_{12} = -\frac{1}{6EI}\cdot 1 \cdot 1 \cdot L + \frac{1}{GA_v}\cdot\left(-\frac{1}{L}\right)\cdot\left(-\frac{1}{L}\right)\cdot L = -\frac{L}{6EI} + \frac{1}{GA_v L} \tag{6}$$

$$f_{22} = \frac{1}{3EI}\cdot 1 \cdot 1 \cdot L + \frac{1}{GA_v}\cdot\left(-\frac{1}{L}\right)\cdot\left(-\frac{1}{L}\right)\cdot L = \frac{L}{3EI} + \frac{1}{GA_v L} \tag{7}$$

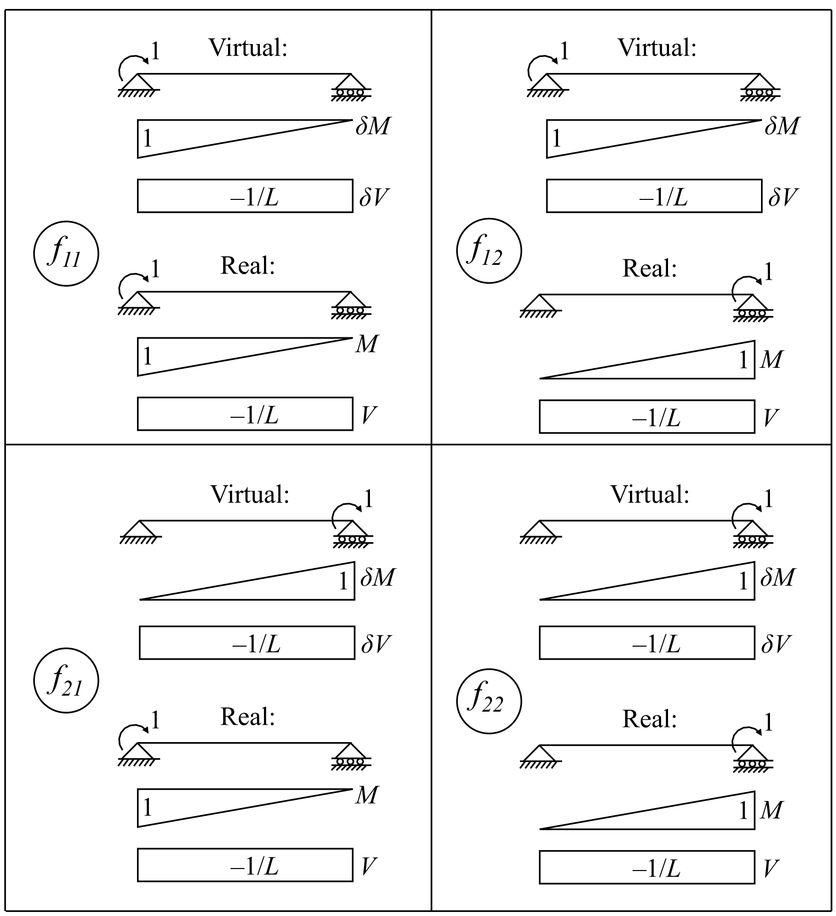

Figure 2: Beam cases for virtual work computations.

The results are summarized in the equation

$$\begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} \tag{8}$$

By defining the auxiliary coefficient

$$\alpha = \frac{12EI}{GA_v L^2} \tag{9}$$

and extracting the results of Figure 2, the flexibility matrix in the basic configuration is:

$$\begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \frac{L}{6EI} \cdot \begin{bmatrix} \left(2 + \frac{\alpha}{2}\right) & -\left(1 - \frac{\alpha}{2}\right) \\ -\left(1 - \frac{\alpha}{2}\right) & \left(2 + \frac{\alpha}{2}\right) \end{bmatrix} \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} \tag{10}$$

In passing, it is noted that the expression for a for a rectangular cross-section with zero Poisson's ratio is

$$\alpha = \frac{12 \cdot E \cdot \frac{b \cdot h^3}{12}}{\frac{E}{2} \cdot \frac{5}{6} \cdot b \cdot h \cdot L^2} = \frac{12}{5} \cdot \left(\frac{h}{L}\right)^2 \tag{11}$$

which reasonably suggests that beams with higher h/L ratio exhibits more shear deformation. Next, inversion of the flexibility matrix in Eq. (10) yields the stiffness matrix in the basic configuration, modified with shear deformation:

$$\mathbf{K}_b = \frac{EI}{(1+\alpha)L} \cdot \begin{bmatrix} (4+\alpha) & (2-\alpha) \\ (2-\alpha) & (4+\alpha) \end{bmatrix} \tag{12}$$

Transformation to the local coordinate system with the transformation $\mathbf{K}_l = \mathbf{T}_{bl}^T \mathbf{K}_b \mathbf{T}_{bl}$ shown in the document on the computational stiffness method yields:

$$\mathbf{K}_l = \frac{1}{(1+\alpha)} \cdot \begin{bmatrix} \frac{12EI}{L^3} & -\frac{6EI}{L^2} & -\frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ -\frac{6EI}{L^2} & \frac{(4+\alpha) \cdot EI}{L} & \frac{6EI}{L^2} & \frac{(2-\alpha) \cdot EI}{L} \\ -\frac{12EI}{L^3} & \frac{6EI}{L^2} & \frac{12EI}{L^3} & \frac{6EI}{L^2} \\ -\frac{6EI}{L^2} & \frac{(2-\alpha) \cdot EI}{L} & \frac{6EI}{L^2} & \frac{(4+\alpha) \cdot EI}{L} \end{bmatrix} \tag{13}$$

To ease the extraction of values from Eq. (13) in hand calculations, the stiffness coefficients in Eq. (13) are provided for two fundamental beam cases in Figure 3.

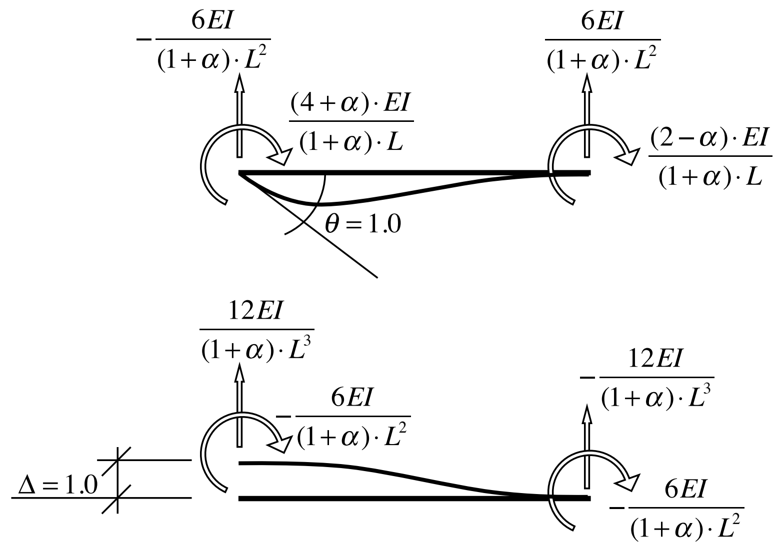

Figure 3: Amendment of fundamental beam cases with shear deformation terms.

Principle of Virtual Displacements

This principle represents the finite element approach, in which shape functions are substituted into the weak form of the boundary value problem. To this end, consider the principle $\delta W_{int}=\delta W_{ext}$ including the work associated with shear deformation:

$$\int_V \sigma\,\delta\varepsilon\,dV + \int_V \tau\,\delta\gamma\,dV = \int_0^L q_z\,\delta w\,dx \tag{14}$$

In the following, the right-hand side of Eq. (14), i.e., the external work is neglected because it remains the same as in the regular Euler-Bernoulli theory. In regards to the left-hand side, equations for material law and kinematics are needed. The two-part material law reads

$$\sigma = E\cdot\varepsilon \tag{15}$$

and

$$\tau = G\cdot\gamma \tag{16}$$

where τ=uniform shear stress over the shear area A_v, γ=shear angle for entire cross-section, and $G=E/(2(1+\nu))$. In Euler-Bernoulli beam elements there is only one unknown displacement field along the beam: $w(x)$. When formulating the equations for kinematic compatibility for Timoshenko beam elements it is important to recognize that there are two different deformation fields: $w(x)$ and $\theta(x)$, where θ is the total rotation of the cross-section. Importantly,

$$\theta \neq \frac{dw}{dx} \tag{17}$$

In other words, in Timoshenko beam theory the section remain plane but not perpendicular to the neutral axis.

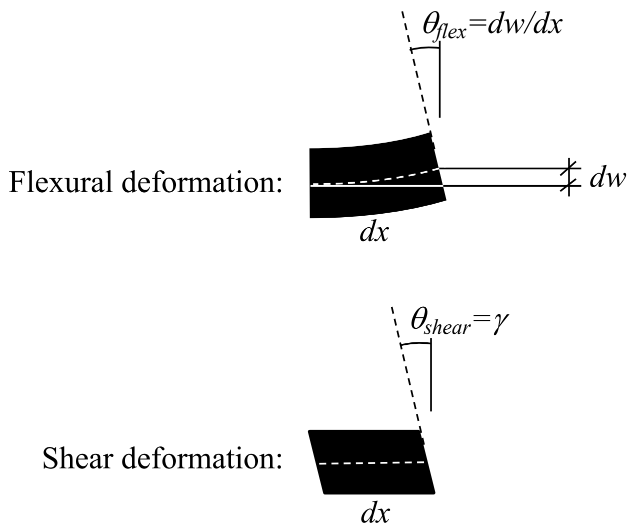

Figure 4: Total rotation of a cross-section in Timoshenko beam theory.

The two parts of the total cross-section rotation, θ, are shown in Figure 4 and expressed as

$$\theta = \frac{dw}{dx} + \gamma \tag{18}$$

Eq. (18) yields the shear strain sought in Eq. (14):

$$\gamma = \theta - \frac{dw}{dx} \equiv \theta - w' \tag{19}$$

The axial strain is

$$\varepsilon = \frac{du}{dx} = -z \cdot \frac{d\theta}{dx} \equiv -z \cdot \theta' \tag{20}$$

The material law and kinematic equations are now substituted in Eq. (14), yielding

$$\int_V E(-z \cdot \theta')(-z \cdot \delta\theta')\, dV + \int_V G(\theta - w')(\delta\theta - \delta w')\, dV = \int_0^L q_z\, \delta w\, dx \tag{21}$$

Following the finite element method, shape functions are now employed to discretize the unknown field functions, here both $w(x)$ and $\theta(x)$:

$$w = \mathbf{N}_w \mathbf{u} = \begin{Bmatrix} N_1 & 0 & N_2 & 0 \end{Bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} \tag{22}$$

$$\theta = \mathbf{N}_\theta \mathbf{u} = \begin{Bmatrix} 0 & N_1 & 0 & N_2 \end{Bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} \tag{23}$$

where N_1=1–x/L and N_2=x/L and the degree of freedom numbering is repeated in Figure 5 for convenience.

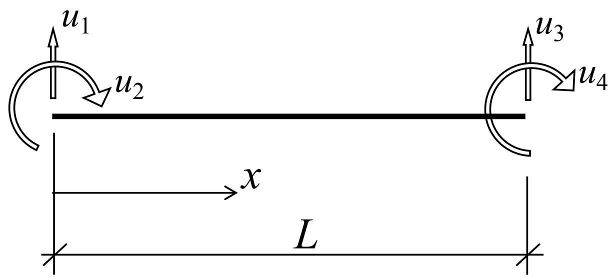

Figure 5: Degrees of freedom in the local element configuration.

Using the same discretization for both real and virtual deformation fields, i.e., substituting Eqs. (22) and (23) into Eq. (21) yields, after defining I and rearranging terms:

$$\delta \mathbf{u}^T \left(\left(\int_0^L EI \left(\mathbf{N}_\theta'\right)^T \left(\mathbf{N}_\theta'\right) dx + \int_0^L GA_v \left(\mathbf{N}_\theta - \mathbf{N}_w'\right)^T \left(\mathbf{N}_\theta - \mathbf{N}_w'\right) dx \right) \mathbf{u} - \int_0^L q_z \mathbf{N}_w^T \, dx \right) = 0 \tag{24}$$

The requirement that the virtual displacement be arbitrary yields the follow system of equations

$$\mathbf{K}_{flex} + \mathbf{K}_{shear} = \mathbf{F} \tag{25}$$

where

$$\mathbf{K}_{flex} = \int_0^L EI \left(\mathbf{N}_\theta'\right)^T \left(\mathbf{N}_\theta'\right) dx \tag{26}$$

$$\mathbf{K}_{shear} = \int_0^L GA_v \left(\mathbf{N}_\theta - \mathbf{N}_w'\right)^T \left(\mathbf{N}_\theta - \mathbf{N}_w'\right) dx \tag{27}$$

$$\mathbf{F} = \int_0^L q_z \mathbf{N}_w^T \, dx \tag{28}$$

Substitution of shape functions and integration yields

$$\mathbf{K}_{flex} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{EI}{L} & 0 & -\frac{EI}{L} \\ 0 & 0 & 0 & 0 \\ 0 & -\frac{EI}{L} & 0 & \frac{EI}{L} \end{bmatrix} \tag{29}$$

$$\mathbf{K}_{shear} = \begin{bmatrix} \frac{GA_v}{L} & \frac{GA_v}{2} & -\frac{GA_v}{L} & \frac{GA_v}{2} \\ \frac{GA_v}{2} & \frac{GA_v \cdot L}{3} & -\frac{GA_v}{2} & \frac{GA_v \cdot L}{6} \\ -\frac{GA_v}{L} & -\frac{GA_v}{2} & \frac{GA_v}{L} & -\frac{GA_v}{2} \\ \frac{GA_v}{2} & \frac{GA_v \cdot L}{6} & -\frac{GA_v}{2} & \frac{GA_v \cdot L}{3} \end{bmatrix} \tag{30}$$

$$\mathbf{K}_{total} = \mathbf{K}_{flex} + \mathbf{K}_{shear} = \begin{bmatrix} \frac{GA_v}{L} & \frac{GA_v}{2} & -\frac{GA_v}{L} & \frac{GA_v}{2} \\ \frac{GA_v}{2} & \frac{EI}{L} + \frac{GA_v \cdot L}{3} & -\frac{GA_v}{2} & -\frac{EI}{L} + \frac{GA_v \cdot L}{6} \\ -\frac{GA_v}{L} & -\frac{GA_v}{2} & \frac{GA_v}{L} & -\frac{GA_v}{2} \\ \frac{GA_v}{2} & -\frac{EI}{L} + \frac{GA_v \cdot L}{6} & -\frac{GA_v}{2} & \frac{EI}{L} + \frac{GA_v \cdot L}{3} \end{bmatrix} \tag{31}$$

$$\mathbf{F} = \begin{Bmatrix} \frac{q_z L}{2} \\ 0 \\ \frac{q_z L}{2} \\ 0 \end{Bmatrix} \tag{32}$$

The stiffness matrix in Eq. (31) is disappointing compared with the result obtained in Eq. (13). In fact, Eq. (13) is far more accurate. Why is the finite element method disappointing us here? The reason is that we have *two* deformation fields to interpolate, w and θ, with only four degrees of freedom. The result is that the shape functions, two for w and two for θ, are too coarse to give good results. The conclusion is: use Eq. (13) or increase the number of degrees of freedom for the element; adding degrees of freedom somewhere between the two ends is one option. In closing it is noted that, paradoxically, for plate elements the inclusion of shear deformation makes it easier, not harder, to establish the stiffness matrix:

- Beam elements
    - Euler-Bernoulli theory (thin beams): Finite element approach works well
    - Timoshenko theory (deep beams): Finite element approach is harder
- Plate elements
    - Kirchhoff theory (thin plates): Finite element approach is harder
    - Mindlin theory (thick plates): Finite element approach works well