Shallow Truss Snap-through

A classical demonstration of geometrical nonlinearity is the "snap-through" problem illustrated in two versions in Figure 1. This is essentially a "stick model," so there is no bending of the member(s). It is instructive to solve the snap-through problem analytically. To that end, the governing equations for this boundary value problem are established in the following.

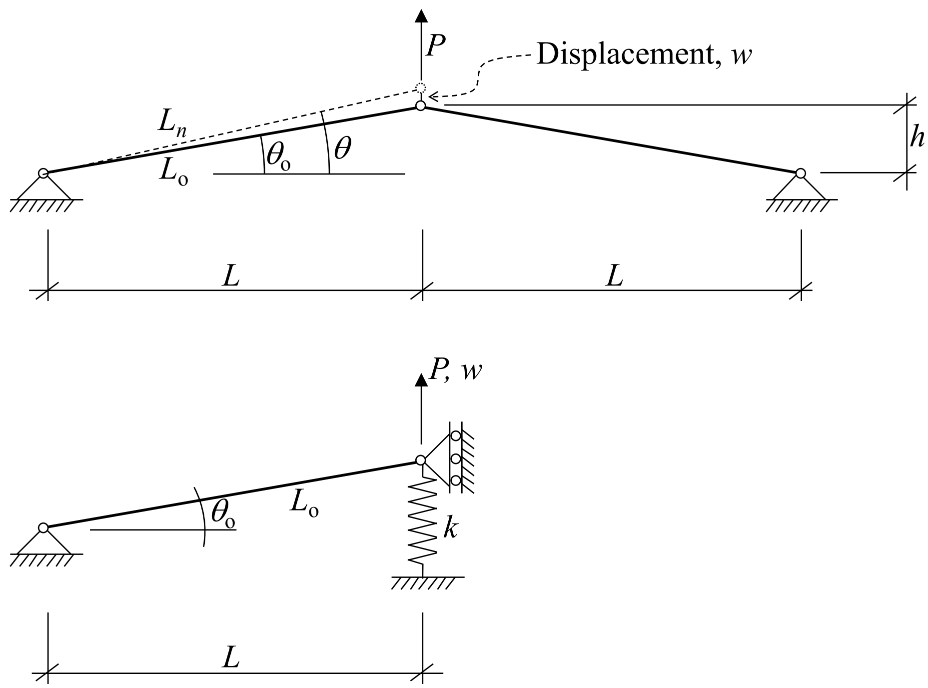

Figure 1: When *P* is applied in the downwards direction the snap-through phenomenon is observed.

The basic linear material law is used:

$$\sigma = E \cdot \varepsilon \tag{1}$$

The stress resultant equation is also from basic structural analysis:

$$N = \sigma \cdot A \tag{2}$$

The kinematic equation introduces geometric nonlinearity. The force presses the structure downwards the length of the inclined member changes. That change in length, divided by the original length, is the "engineering strain:"

$$\varepsilon = \frac{L_n - L_o}{L_o} = \frac{L_n}{L_o} - 1 = \frac{\sqrt{L^2 + (h+w)^2}}{\sqrt{L^2 + h^2}} - 1 \tag{3}$$

Multiplying through by L/L yields

$$\varepsilon = \frac{\sqrt{1+\left(\frac{h+w}{L}\right)^2}}{\sqrt{1+\left(\frac{h}{L}\right)^2}} - 1 \tag{4}$$

Now employing the Taylor series approximations

$$\sqrt{1+x^2} \approx 1 + \frac{1}{2}\cdot x^2 \quad \text{and} \quad \frac{1}{\sqrt{1+x^2}} \approx 1 - \frac{1}{2}\cdot x^2 \tag{5}$$

yields

$$\begin{aligned} \varepsilon &= \left(1+\frac{1}{2}\cdot\left(\frac{h+w}{L}\right)^2\right)\cdot\left(1-\frac{1}{2}\cdot\left(\frac{h}{L}\right)^2\right)-1 \\ &= \left(1+\frac{1}{2}\cdot\left(\frac{w}{L}+\frac{h}{L}\right)^2\right)\cdot\left(1-\frac{1}{2}\cdot\left(\frac{h}{L}\right)^2\right)-1 \end{aligned} \tag{6}$$

Multiplying out the parentheses yields

$$\varepsilon = \frac{w}{L}\cdot\frac{h}{L} + \frac{1}{2}\cdot\left(\frac{w}{L}\right)^2 - \frac{1}{4}\cdot\cancel{\left(\frac{w}{L}\right)^2\cdot\left(\frac{h}{L}\right)^2} - \frac{1}{2}\cdot\cancel{\frac{w}{L}\cdot\left(\frac{h}{L}\right)^3} - \frac{1}{4}\cdot\cancel{\left(\frac{h}{L}\right)^4} \tag{7}$$

where the higher-order terms are cancelled. That means the strain can be written as a linear plus a nonlinear displacement term:

$$\varepsilon = \frac{h}{L}\cdot\gamma + \frac{1}{2}\cdot\gamma^2 \tag{8}$$

where the displacement is expressed as

$$\gamma \equiv \frac{w}{L} \tag{9}$$

The equilibrium equation is established in the deformed configuration:

$$P = N\cdot\sin(\theta) + k\cdot w \tag{10}$$

where N is positive in tension and the bottom case in Figure 1 is considered. The top case is slightly different; in that case the term $k\cdot w$ is dropped and $N\cdot\sin(\theta)$ is multiplied by 2. The equilibrium equation is simplified by expressing $\sin(\theta)$ as

$$\sin(\theta) = \frac{h+w}{L_o} = \frac{h+w}{\sqrt{L^2+h^2}} \tag{11}$$

Multiplying by L/L yields

$$\sin(\theta) = \frac{\frac{h+w}{L}}{\sqrt{1+\left(\frac{h}{L}\right)^2}} \tag{12}$$

Use of Eqs. (5) and (9) yields

$$\sin(\theta) = \left(\frac{h}{L}+\gamma\right)\cdot\left(1-\frac{1}{2}\cdot\left(\frac{h}{L}\right)^2\right) \tag{13}$$

which means the equilibrium equation reads

$$P = N\cdot\left(\frac{h}{L}+\gamma\right)\cdot\left(1-\frac{1}{2}\cdot\left(\frac{h}{L}\right)^2\right)+k\cdot w \tag{14}$$

Combining all the governing equations yields

$$\begin{aligned} P &= N\cdot\left(\frac{h}{L}+\gamma\right)\cdot\left(1-\frac{1}{2}\cdot\left(\frac{h}{L}\right)^2\right)+k\cdot w \\ &= A\cdot\sigma\cdot\left(\frac{h}{L}+\gamma\right)\cdot\left(1-\frac{1}{2}\cdot\left(\frac{h}{L}\right)^2\right)+k\cdot w \\ &= EA\cdot\varepsilon\cdot\left(\frac{h}{L}+\gamma\right)\cdot\left(1-\frac{1}{2}\cdot\left(\frac{h}{L}\right)^2\right)+k\cdot w \\ &= EA\cdot\left(\frac{h}{L}\cdot\gamma+\frac{1}{2}\cdot\gamma^2\right)\cdot\left(\frac{h}{L}+\gamma\right)\cdot\left(1-\frac{1}{2}\cdot\left(\frac{h}{L}\right)^2\right)+k\cdot w \end{aligned} \tag{15}$$

When h is relatively small compared with L, i.e., a "shallow truss" situation, then the last of the three parentheses is approximately equal to unity and the remaining expression is a third order polynomial expression with respect to the displacement

$$P = EA\cdot\left(\left(\frac{h}{L}\right)^2\cdot\gamma+\frac{3}{2}\cdot\frac{h}{L}\cdot\gamma^2+\frac{1}{2}\cdot\gamma^3\right)+k\cdot L\cdot\gamma \tag{16}$$

Eq. (16) is plotted in Figure 2 for E=200,000N/mm^2, A=b^2, b=20mm, L=2,000mm, and h=50mm. Negative (downwards) displacement is applied to observe the snap-through behaviour. Notice how the force increases at first, but then decreases to zero when the displacement reaches h; at that point the snap-through occurs and the force changes sign. From that point the truss element remains firmly below the horizontal line. Ultimately the force will keep increasing, as the element is pushed further and further downwards.

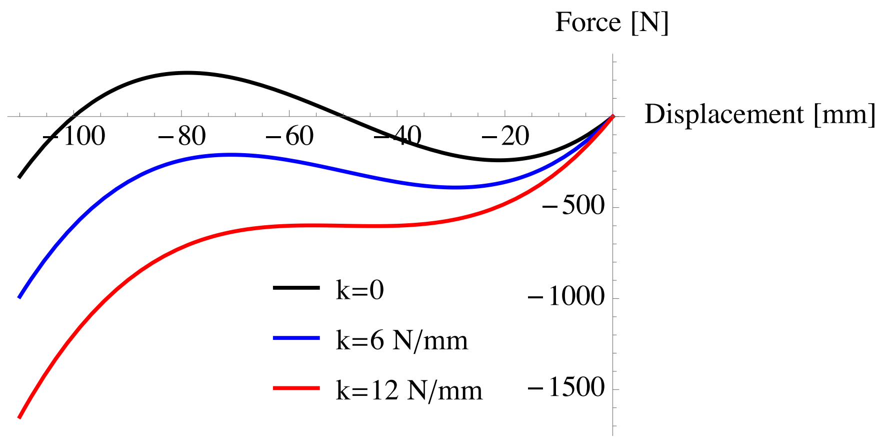

Figure 2: Force-displacement relationship for the snap-through problem.