Numerical Integration (Quadrature)

Quadrature is another name for numerical integration, which is helpful in these notes for evaluating integrals in the finite element method. We consider integrals of the type

$$\int_a^b f(x)\,dx \tag{1}$$

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2} f(x,y)\,dx\,dy \tag{2}$$

and

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} f(x,y,z)\,dx\,dy\,dz \tag{3}$$

i.e., single-fold, two-fold, and three-fold integrals. To apply quadrature rules, all such integrals are transformed into a domain from –1 to 1, in all directions. To transform the integral in Eq. (1) into an integral along the ξ-axis from –1 to 1, the transformation is

$$x = \left(\frac{b-a}{2}\right)\cdot\xi + \left(\frac{b+a}{2}\right) \tag{4}$$

In addition to substituting Eq. (4) into the function $f(x)$, it is necessary to transform the integral differentials. To accomplish that, the integrand is multiplied by the determinant of the Jacobian matrix, i.e., $J=|\mathbf{J}|$. The Jacobian scalar or matrix contains the ratio of differentials in the different coordinate systems. For the integrals shown above, the Jacobian scalar/matrices read

$$J = \frac{dx}{d\xi} = \frac{b-a}{2} \tag{5}$$

$$J = \begin{vmatrix} \left(\frac{b_1-a_1}{2}\right) & 0 \\ 0 & \left(\frac{b_2-a_2}{2}\right) \end{vmatrix} = \left(\frac{b_1-a_1}{2}\right)\cdot\left(\frac{b_2-a_2}{2}\right) \tag{6}$$

$$J = \begin{vmatrix} \left(\frac{b_1 - a_1}{2}\right) & 0 & 0 \\ 0 & \left(\frac{b_2 - a_2}{2}\right) & 0 \\ 0 & 0 & \left(\frac{b_3 - a_3}{2}\right) \end{vmatrix} = \left(\frac{b_1 - a_1}{2}\right) \cdot \left(\frac{b_2 - a_2}{2}\right) \cdot \left(\frac{b_3 - a_3}{2}\right) \quad (7)$$

The Gauss family of integration rules is important in structural analysis. Consider the transformed single-fold integral in Eq. (1). The integration rule is written

$$\int_{-1}^{1} f(\xi)\, d\xi = \sum_{i=1}^{N} w_i\, f(\xi_i) \quad (8)$$

where N=number of integration points, w_i=integration weights, and ξ_i=integration points. The integration weights & points are derived by looking at reference cases with known solutions. First consider N=1, in which case Gauss integration reads

$$\int_{-1}^{1} f(\xi)\, d\xi = w_1\, f(\xi_1) \quad (9)$$

With two unknowns, i.e., w_1 and ξ_1, we can integrate linear functions, $f(\xi)=a\xi+b$, which has two constants, in an exact manner. That means the exact solution

$$\int_{-1}^{1} (a\xi + b)\, d\xi = 2b \quad (10)$$

should equal the quadrature solution

$$w_1\,(a\xi_1 + b) = a w_1 \xi_1 + w_1 b \quad (11)$$

The only way for that to be the case for any a and b is that w_1=2 and $w_1\xi_1$=0, meaning that ξ_1=0. Now consider N=2, in which case we have the four unknowns ξ_1, ξ_2, w_1, and w_2. With four parameters, we can integrate a 3rd order polynomial, which has four constants, in an exact manner. That means the exact solution

$$\int_{-1}^{1} (a\xi^3 + b\xi^2 + c\xi + d)\, d\xi = \frac{2}{3} b + 2d \quad (12)$$

should equal the quadrature solution

$$w_1\,(a\xi_1^3 + b\xi_1^2 + c\xi_1 + d) + w_2\,(a\xi_2^3 + b\xi_2^2 + c\xi_2 + d) \quad (13)$$

Collecting terms in a, b, c, and d yields four equations by comparing Eq. (12) with Eq. (13):

Terms with *a*: $$\xi_1^3 w_1 + \xi_2^3 w_2 = 0 \tag{14}$$

Terms with *b*: $$\xi_1^2 w_1 + \xi_2^2 w_2 = \frac{2}{3} \tag{15}$$

Terms with *c*: $$\xi_1 w_1 + \xi_2 w_2 = 0 \tag{16}$$

Terms with *d*: $$w_1 + w_2 = 2 \tag{17}$$

Solving those four equations for the four unknowns yields $\xi_1 = -\frac{1}{\sqrt{3}} = -0.57735$, $\xi_2 = \frac{1}{\sqrt{3}} = 0.57735$, and $w_1 = w_2 = 1$. Table 1 provides the integration points & weights for N up to four. It is understood that Gauss integration provides exact results for integration of polynomials of order up to $2N–1$. Two and three-fold integrals are evaluated by applying the same points and weights in orthogonal directions. For example, for a two-fold integral:

$$\int_{-1}^{1}\int_{-1}^{1} f(\xi,\eta)\, d\xi\, d\eta = \sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j(\xi_i,\eta_j) \tag{18}$$

Table 1: Integration points and weights for Gauss quadrature.

N	x_i	w_i
1	0	2
2	–0.577350269189626 +0.577350269189626	1 1
3	–0.774596669241483 0 +0.774596669241483	0.555555555555556 0.888888888888889 0.555555555555556
4	–0.861136311594053 –0.339981043584856 +0.339981043584856 +0.861136311594053	0.347854845137454 0.652145154862546 0.652145154862546 0.347854845137454