Frame Elements with Stability Functions

The document on Beams with Axial Force establishes the following differential equation for a beam element by considering equilibrium in its deformed configuration:

$$EI \cdot w'''' + P \cdot w'' = 0 \tag{1}$$

Thereafter, in the document on Frame Elements with P-Delta, the geometric stiffness matrix is established using polynomial shape functions. That is an approximation, because the exact solution to the differential equation in Eq. (1) does not contain polynomial functions. In the present document, the exact solution is explored. Considering the element in its Local configuration, shown in Figure 1, the first column of the stiffness matrix is obtained by setting u_1=1, u_2=u_3=u_4=0, and computing the force along each degree of freedom:

$$\begin{aligned} F_1 &= V(0) = EI \cdot w'''(0) \\ F_2 &= M(0) = EI \cdot w''(0) \\ F_3 &= -V(L) = -EI \cdot w'''(L) \\ F_4 &= -M(L) = -EI \cdot w''(L) \end{aligned} \tag{2}$$

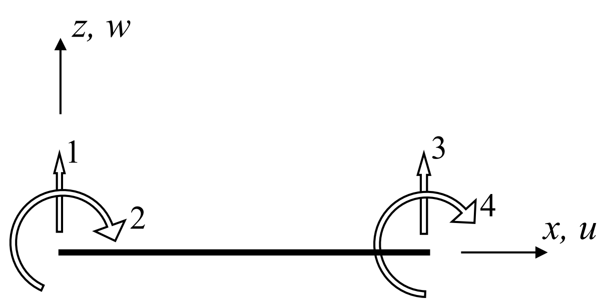

Figure 1: Beam in the local element configuration.

To simplify the resulting expressions, R.K. Livesley introduced the following auxiliary functions in his book entitled Matrix Methods of Structural Analysis published by Pergamon Press in Oxford, UK in 1975:

$$\begin{aligned}
\phi_1 &= \frac{\beta}{\tan(\beta)} \\
\phi_2 &= \frac{1}{3} \cdot \frac{\beta^2}{1-\phi_1} \\
\phi_3 &= \frac{1}{4} \cdot \phi_1 + \frac{3}{4} \cdot \phi_2 \\
\phi_4 &= -\frac{1}{2} \cdot \phi_1 + \frac{3}{2} \cdot \phi_2 \\
\phi_5 &= \phi_1 \cdot \phi_2
\end{aligned} \tag{3}$$

where

$$\beta = \frac{L}{2} \cdot \sqrt{\frac{P}{EI}} \tag{4}$$

That leads to the following exact stiffness matrix, including both elastic stiffness and geometric stiffness contributions:

$$\mathbf{K} = \frac{2EI}{L^3} \cdot \begin{bmatrix} 6\phi_5 & -3L\phi_2 & -6\phi_5 & -3L\phi_2 \\ -3L\phi_2 & 2L^2\phi_3 & 3L\phi_2 & L^2\phi_4 \\ -6\phi_5 & 3L\phi_2 & 6\phi_5 & 3L\phi_2 \\ -3L\phi_2 & L^2\phi_4 & -3L\phi_2 & 2L^2\phi_3 \end{bmatrix} \tag{5}$$

That stiffness matrix is exact, but comprises a mix of the elastic and the geometric stiffness matrix. That means the eigenvalue problem for buckling loads/modes does not emanate from this formulation. That eigenvalue problem has the elastic and geometric stiffness matrices separate, addressed in the document on Frame Elements with P-Delta.