# Frame Elements with P-Delta

The objective in this document is to include the effect of axial force, $P$, on the lateral stiffness of a beam that bends. Specifically, we wish to modify the ordinary stiffness matrix for beam bending to include P-delta effects. The correction in stiffness compared with the ordinary stiffness matrix is called "geometric stiffness." To accomplish the stated objective, it is necessary to consider the beam element in its local configuration, shown in Figure 1. The degrees of freedom in the basic configuration are not capable of describing the sufficient amount of lateral displacement, delta, to fully account for the P-delta effects.

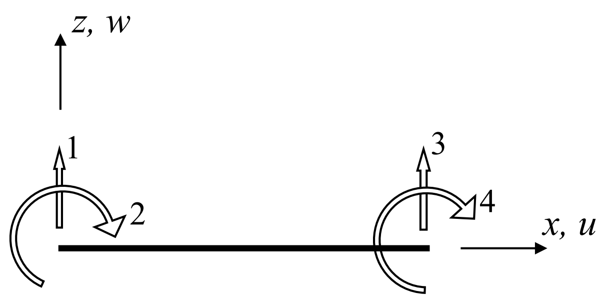


**Figure 1: Beam in the local element configuration.**

Consider the differential equation for beam bending, amended with the P-delta effect. In the absence of distributed load $q$, the weighted and integrated version reads

$$\int_0^L \left( EI \cdot w'''' + P \cdot w'' \right) \delta w \, dx = 0 \tag{1}$$

Integration by parts and cancelling boundary terms yield

$$\int_0^L EI \cdot w'' \delta w'' dx - \int_0^L P \cdot w' \delta w' dx = 0 \tag{2}$$

Discretization of the real and virtual displacement fields by shape functions, i.e., $w(x)=\mathbf{Nu}$ and $\delta w(x)=\mathbf{N}\delta\mathbf{u}$, where the vector $\mathbf{u}$ collects the displacements along the degrees of freedom, yields

$$\left( \int_0^L EI \cdot \mathbf{N}''^T \cdot \mathbf{N}'' dx - P \cdot \underbrace{\int_0^L \mathbf{N}'^T \cdot \mathbf{N}' dx}_{\mathbf{K}^{\mathrm{G}}} \right) \cdot \mathbf{u} = 0 \tag{3}$$

where the geometric stiffness matrix, $\mathbf{K}^{\mathrm{G}}$, is defined. Two paths can now be followed to obtain the geometric stiffness matrix in the local element configuration. Because the basic configuration neglects rigid-body movement of the element, using the shape functions for that configuration will miss the "P/L terms." In other words, the P-delta effects associated with rigid-body movement of the element will be missed if the integral in Eq. (3) is conducted in the basic element configuration. For that reason, the following third-order polynomial shape functions for the beam element in the *local* configuration are employed:

$$\begin{aligned} N_1(x) &= \frac{2x^3}{L^3} - \frac{3x^2}{L^2} + 1 \\ N_2(x) &= -\frac{x^3}{L^2} + \frac{2x^2}{L} - x \\ N_3(x) &= -\frac{2x^3}{L^3} + \frac{3x^2}{L^2} \\ N_4(x) &= -\frac{x^3}{L^2} + \frac{x^2}{L} \end{aligned} \tag{4}$$

As a result, the following total stiffness matrix is obtained:

$$\mathbf{K} = \begin{bmatrix} \frac{12EI}{L^3} & -\frac{6EI}{L^2} & -\frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ -\frac{6EI}{L^2} & \frac{4EI}{L} & \frac{6EI}{L^2} & \frac{2EI}{L} \\ -\frac{12EI}{L^3} & \frac{6EI}{L^2} & \frac{12EI}{L^3} & \frac{6EI}{L^2} \\ -\frac{6EI}{L^2} & \frac{2EI}{L} & \frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix} - \underbrace{P \cdot \begin{bmatrix} \frac{6}{5L} & -\frac{1}{10} & -\frac{6}{5L} & -\frac{1}{10} \\ -\frac{1}{10} & \frac{2L}{15} & \frac{1}{10} & -\frac{L}{30} \\ -\frac{6}{5L} & \frac{1}{10} & \frac{6}{5L} & \frac{1}{10} \\ -\frac{1}{10} & -\frac{L}{30} & \frac{1}{10} & \frac{2L}{15} \end{bmatrix}}_{\mathbf{K}^{\mathrm{G}}} \tag{5}$$

If the integration had been conducted in the basic configuration, as mentioned above, it would be necessary to add the "P/L terms" to the result:

$$\mathbf{K}^{\mathrm{G}} = P \cdot \begin{bmatrix} \frac{1}{5L} & -\frac{1}{10} & -\frac{1}{5L} & -\frac{1}{10} \\ -\frac{1}{10} & \frac{2L}{15} & \frac{1}{10} & -\frac{L}{30} \\ -\frac{1}{5L} & \frac{1}{10} & \frac{1}{5L} & \frac{1}{10} \\ -\frac{1}{10} & -\frac{L}{30} & \frac{1}{10} & \frac{2L}{15} \end{bmatrix} + \begin{bmatrix} \frac{P}{L} & 0 & -\frac{P}{L} & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{P}{L} & 0 & \frac{P}{L} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{6}$$

The last term in Eq. (6) can be labelled the "Big P-delta" effect, which is all there is for a truss element, while the first term on the right-hand side represents the "Small P-delta" effect associated with bending in frame elements. To ease the extraction of values from Eq. (6) in hand calculations, the stiffness coefficients in Eq. (6) are provided for two fundamental beam cases in Figure 2.

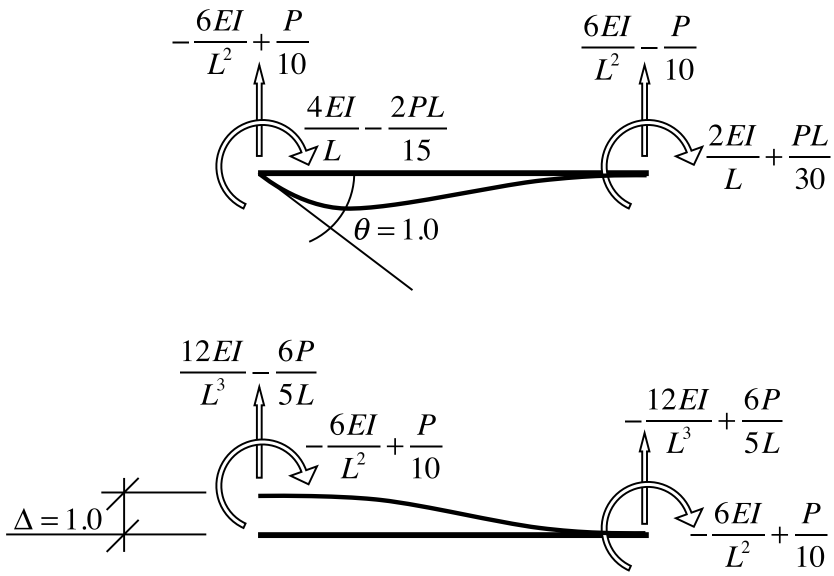


**Figure 2: Amendment of fundamental beam cases with geometric stiffness terms.**

The results presented in this document is essentially a second-order linearized way of including the effect of axial force on lateral stiffness. Another document on this website takes the P-delta and geometric stiffness matrix in this document to the next level, i.e., nonlinear structural analysis with geometric nonlinearity.