Euler-Bernoulli Beam Elements

Euler-Bernoulli theory is a theory that governs the bending of beams whose cross-sections remain plane and perpendicular to the neutral axis during bending. The key objective in this document is to establish the stiffness matrix, **K**, for an Euler-Bernoulli beam element, without axial and torsional degrees of freedom. Figure 1 shows the degrees of freedom (DOFs) in the "basic" and "local" configuration for the considered element.

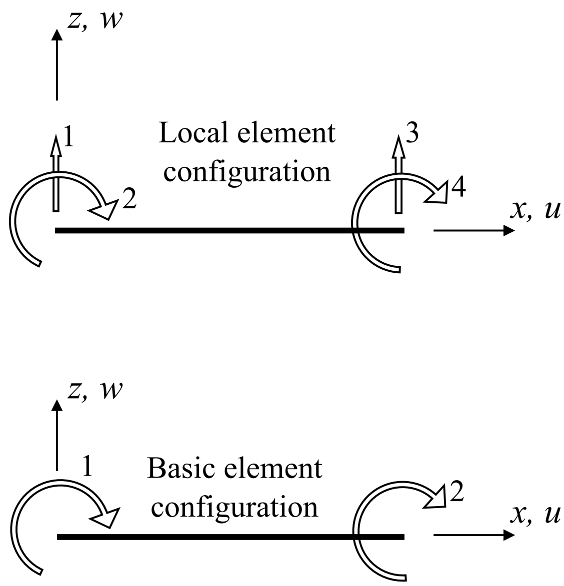

Figure 1: Beam element configurations for 2D structural analysis.

One approach for establishing the stiffness matrix is to apply the slope-deflection equation, which is derived using virtual work in another document on this website, to the basic configuration. Keeping in mind that K_{ij}=force along DOF number i caused by a unit deformation along DOF number j the result is

$$\mathbf{K}_{basic} = \begin{bmatrix} \frac{4EI}{L} & \frac{2EI}{L} \\ \frac{2EI}{L} & \frac{4EI}{L} \end{bmatrix} \tag{1}$$

Applying the transformation matrix $\underline{\mathbf{T}}_{basic\text{-}to\text{-}local}$ from the document that explains the computational stiffness matrix the local element stiffness matrix is

$$\mathbf{K}_{local} = \mathbf{T}_{lb}^T \mathbf{K}_{local} \mathbf{T}_{lb} = \begin{bmatrix} \frac{12EI}{L^3} & -\frac{6EI}{L^2} & -\frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ -\frac{6EI}{L^2} & \frac{4EI}{L} & \frac{6EI}{L^2} & \frac{2EI}{L} \\ -\frac{12EI}{L^3} & \frac{6EI}{L^2} & \frac{12EI}{L^3} & \frac{6EI}{L^2} \\ -\frac{6EI}{L^2} & \frac{2EI}{L} & \frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix} \tag{2}$$

Another approach for establishing the element stiffness matrix is to use the finite element method. The most straightforward way of applying that method in this context is to substitute shape functions into the “weak form” of the boundary value problem (BVP). There are two ways of obtaining the weak form, both shown below.

Weak Form from Virtual Work

The principle of virtual work reads

$$\delta W_{int} = \delta W_{ext} \tag{3}$$

where the subscript “int” is for internal work and the subscript “ext” is for external work. The principle of virtual work comes in two forms, the principle of virtual displacements and the principle of virtual forces. It is the former that is most often applied in the finite element method. For the present case, the principle of virtual displacements reads

$$\int_V \sigma\, \delta\varepsilon\, dV = \int_0^L q_z\, \delta w\, dx \tag{4}$$

Substitution of the linear elastic material law yields

$$\int_V E\varepsilon\, \delta\varepsilon\, dV = \int_0^L q_z\, \delta w\, dx \tag{5}$$

where E is the modulus of elasticity. Substitution of kinematic compatibility for Euler-Bernoulli beams yields

$$\int_V E \cdot z^2 \cdot w''\, \delta w''\, dV = \int_0^L q_z\, \delta w\, dx \tag{6}$$

The separation of the left-hand side into a cross-section integral and a longitudinal integral along the beam yields

$$\int_0^L EI \cdot w''\, \delta w''\, dx = \int_0^L q_z\, \delta w\, dx \tag{7}$$

where the cross-section constant I, i.e., the moment of inertia, has been introduced. Eq. (7) is the weak form of the BVP for beam bending of Euler-Bernoulli beams.

Weak Form from Strong Form

The weak form can also be established by starting with the strong form of the BVP, i.e., the differential equation. For Euler-Bernoulli beams, that equation is written on residual form as follows:

$$EI \cdot w'''' - q = 0 \tag{8}$$

That equation is then multiplied by a "weight function" and integrated over the beam:

$$\int_0^L \left(EI \cdot w'''' - q \right) \cdot \delta w \cdot dx = 0 \tag{9}$$

This means that, instead of demanding equilibrium along every point along the beam, equilibrium is only required "on average" over the entire beam. This is a weaker form of the BVP and may in itself serve as the foundation for the finite element method. However, we proceed to the weak form already derived in Eq. (7) by reorganizing Eq. (9) to read

$$\int_0^L \left(EI \cdot w'''' \cdot \delta w - q \cdot \delta w \right) dx = 0 \tag{10}$$

We now aim to avoid unequal number of derivatives of w and δw in the first term. Twice application of integration by parts yield

$$\cancel{\left[EI \cdot w''' \cdot \delta w \right]_0^L} - \cancel{\left[EI \cdot w'' \cdot \delta w' \right]_0^L} + \int_0^L \left(EI \cdot w'' \cdot \delta w'' - q \cdot \delta w \right) dx = 0 \tag{11}$$

where the boundary terms cancel. More details about why they cancel are presented in the document on Energy Methods. We observe that Eq. (11) matches Eq. (7), which was derived earlier. Notice that equilibrium was not substituted into virtual work formulation; the principle of virtual displacements represents integrated equilibrium.

Shape Functions

Now consider the beam element in its basic configuration, i.e., with two DOFs as shown in Figure 1. The discretization of this problem by means of shape functions follows. Let the clockwise rotation of the left end be denoted u_1 and let the clockwise rotation at the right end be denoted u_2. Maintaining zero displacement at both ends, third-order polynomial shape functions are possible. Consequently, the shape functions are:

$$w(x) = \mathbf{Nu} = \begin{bmatrix} N_1(x) & N_2(x) \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \tag{12}$$

where

$$\begin{aligned} N_1(x) &= -\frac{1}{L^2} x^3 + \frac{2}{L} x^2 - x \\ N_2(x) &= -\frac{1}{L^2} x^3 + \frac{1}{L} x^2 \end{aligned} \tag{13}$$

Substitution into the weak form yields

$$\int_0^L EI \cdot \left(\mathbf{N}''\mathbf{u}\right) \cdot \left(\mathbf{N}''\delta\mathbf{u}\right) \ \mathrm{d}x - \int_0^L q_z \ \left(\mathbf{N}\delta\mathbf{u}\right) \ \mathrm{d}x = 0 \tag{14}$$

where $\delta\mathbf{u}$ is the virtual nodal deformations because the virtual displacements are discretized by the same shape functions as the actual displacements. Taking the transpose of two parentheses that contain scalars, applying the transpose to each vector in the parenthesis, thereby flipping the order of multiplication inside the parenthesis, and finally rearranging yields

$$\delta\mathbf{u}^T\left(\left[\int_0^L EI \cdot \mathbf{N}''^T\mathbf{N}'' \ \mathrm{d}x\right]\mathbf{u} - \int_0^L q_z \ \mathbf{N}^T \ \mathrm{d}x\right) = 0 \tag{15}$$

Furthermore, because the virtual displacements are arbitrary the parenthesis must be zero for this equation to be generally valid. Consequently, it is rewritten

$$\underbrace{\left[\int_0^L EI \cdot \mathbf{N}''^T \, \mathbf{N}'' dx\right]}_{\text{Stiffness matrix, } \mathbf{K}}\mathbf{u} = \underbrace{\int_0^L q_z \mathbf{N} \, dx}_{\text{Load vector, } \mathbf{F}} \tag{16}$$

where the stiffness matrix and load vector are identified. Notice that this load vector is usually labelled $\bar{\mathbf{F}}$ in the documents on this website. Importantly, the finite element method yields integral expressions for the stiffness matrix and the load vector. Substitution of Eq. (13) into Eq. (16) and assuming that the distributed element load is uniform yields

$$\begin{bmatrix} \frac{4EI}{L} & \frac{2EI}{L} \\ \frac{2EI}{L} & \frac{4EI}{L} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} -\frac{q_z L^2}{12} \\ \frac{q_z L^2}{12} \end{Bmatrix} \tag{17}$$

This stiffness matrix is equal to that of the classical stiffness method because the third-order polynomial shape functions match the solution of the differential equation for beam bending. Under such circumstances the finite element method is exact. Notice that the right-hand side load vector has the correct sign for upward-acting q_z and clockwise rotation DOFs.