Buckling Modes for Two-storey Frame

When the stiffness matrix in linear static structural analysis is amended with geometric stiffness terms it is possible to compute the buckling loads, and associated displaced shapes, i.e., buckling modes, of the structure. There will be as many buckling loads as there are DOFs, but only the smallest is relevant in most practical applications because it is the governing buckling load. The geometric stiffness matrix for each element type is established in separate documents. Upon assembling the final structural stiffness matrix, including geometric stiffness contributions, the system of equilibrium equations is written

$$\left(\mathbf{K} - P\,\mathbf{K}^{\mathrm{G}}\right)\mathbf{u} = \mathbf{F}$$

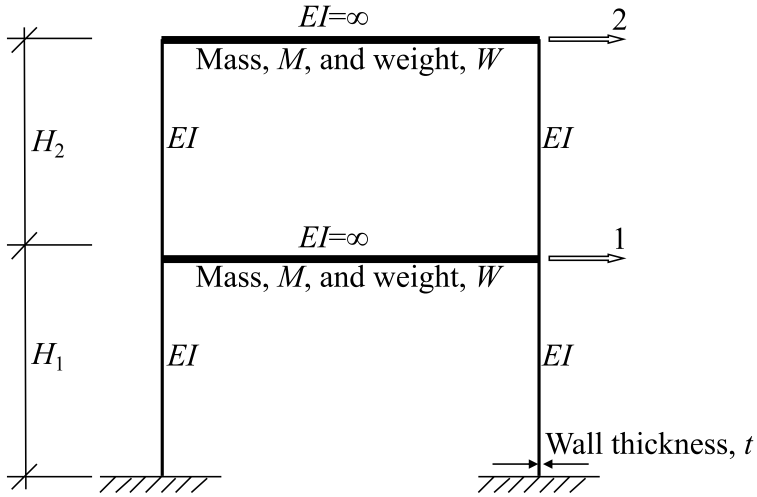
where $\mathbf{K}^{\mathrm{G}}$ is the geometric stiffness matric, not to be confused with the notation for the stiffness matrix in the global element configuration. The factorization of the axial force level P as a multiplier of the geometric stiffness matrix of the structure is noted. This is possible when the considered axial forces are from one source, such as gravity. It is also necessary that the formulation be based on the second-order linearized theory. The use of exact Livesley functions, i.e., the exact solution of the differential equation, prevents the form of the equation above. However, when that form is possible, then the buckling loads, P_{cr}, of the system can be computed. First, remove the other loads that are represented by the external load vector, i.e., set **F**=0. Next, recognize that the remaining system of equations is an eigenvalue problem; it is homogeneous with non-trivial solutions only when the determinant of the coefficient matrix is zero. Hence, the equation

$$\det\left(\mathbf{K} - P\,\mathbf{K}^{\mathrm{G}}\right) = 0$$

is solved to obtain the critical values of the axial load level, which are the buckling loads. Each buckling load has a corresponding buckling mode. While the buckling loads are the eigenvalues, the buckling modes are the eigenvectors. The modes represent the displaced shape of the structure when it buckles at the corresponding buckling load. The amplitude of the deformation is not uniquely determined. However, the shape is obtained by setting one component of **u** equal to unity and solving for the others. Software applications that solve eigenvalue problems do this automatically.

Problem Definition

Consider the frame in the figure below. The floors are considered infinitely rigid. As a result, there are only two degrees of freedom, as shown. The out-of-plane length is 1m. The total mass of each 1m strip of floor is denoted M. The corresponding weight, i.e., 9.81 m/s^2 times the mass is denoted W. The objective is to find the buckling loads and associated buckling modes.



Input [N, m, kg, sec]

```
E = 60 × 10^9;
t = 0.2;
H1 = 4.0;
H2 = 3.5;
M = 10 000;
```

Bending stiffness:

$$EI = E\frac{t^3}{12}$$

which yields: $4. \times 10^7$

Elastic stiffness matrix

$$K0 = \left\{\left\{2\frac{12\,EI}{H1^3} + 2\frac{12\,EI}{H2^3}, -2\frac{12\,EI}{H2^3}\right\}, \left\{-2\frac{12\,EI}{H2^3}, 2\frac{12\,EI}{H2^3}\right\}\right\};$$

```
K0 // MatrixForm
```

which yields: $\begin{pmatrix} 3.73907\times10^7 & -2.23907\times10^7 \\ -2.23907\times10^7 & 2.23907\times10^7 \end{pmatrix}$

Geometric stiffness matrix

$$\text{KG} = \left\{\left\{2\,\frac{6\,(\text{W})}{5\,\text{H1}} + 2\,\frac{6\left(\frac{\text{W}}{2}\right)}{5\,\text{H2}},\ -2\,\frac{6\left(\frac{\text{W}}{2}\right)}{5\,\text{H2}}\right\},\ \left\{-2\,\frac{6\left(\frac{\text{W}}{2}\right)}{5\,\text{H2}},\ 2\,\frac{6\left(\frac{\text{W}}{2}\right)}{5\,\text{H2}}\right\}\right\};$$

```
KG // MatrixForm
```

which yields: $\begin{pmatrix} 0.942857\,\text{W} & -0.342857\,\text{W} \\ -0.342857\,\text{W} & 0.342857\,\text{W} \end{pmatrix}$

Buckling loads & mode shapes

Compute the determinant:

```
det = Det[K0 - KG] // FullSimplify // Expand
```

which yields: $3.3586 \times 10^{14} - 1.85773 \times 10^{7}\,\text{W} + 0.205714\,\text{W}^2$

Then set the determinant equal to zero, to find the buckling loads:

```
soln = Solve[det == 0, W]
```

which yields: $\{\{\text{W} \to 2.5 \times 10^{7}\},\ \{\text{W} \to 6.53061 \times 10^{7}\}\}$

The buckling mode shape associated with the first buckling load is obtained by substituting that buckling load, and setting one of the components of the displacement vector equal to one, as a reference value when solving for the other:

```
Solve[((K0 - KG) /. soln[[1, 1]]).{u1, 1} == {0, 0}, u1]
```

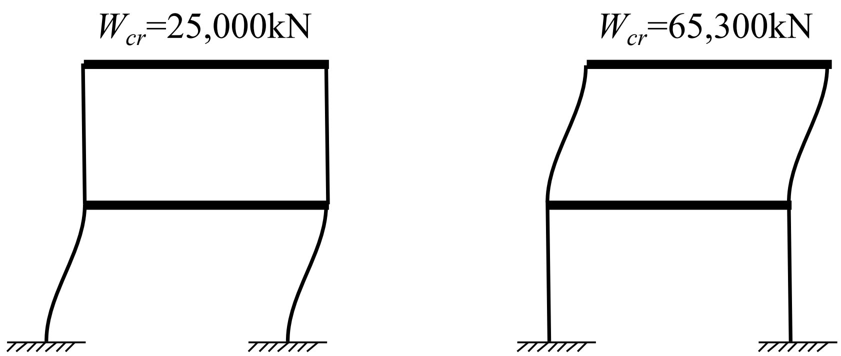
which yields: $\{\{\text{u1} \to 1.\}\}$

Second mode shape:

```
Solve[((K0 - KG) /. soln[[2, 1]]).{u1, 1} == {0, 0}, u1]
```

which yields: $\{\{\text{u1} \to -1.54042 \times 10^{-16}\}\}$

A plot of those two shapes:



Check of eigenvalues, using a function in *Mathematica*:

$$\mathrm{Eigenvalues}\left[\left\{\mathrm{K0},\ \frac{\mathrm{KG}}{\mathrm{W}}\right\}\right]$$

which yields: $\left\{6.53061\times 10^{7},\ 2.5\times 10^{7}\right\}$

Check of eigenvectors, using a function in *Mathematica* (different numbers, but that is fine; the shapes are the same as above):

$$\mathrm{Eigenvectors}\left[\left\{\mathrm{K0},\ \frac{\mathrm{KG}}{\mathrm{W}}\right\}\right]$$

which yields: $\left\{\left\{1.86227\times 10^{-17},\ 1.\right\},\ \{-0.707107,\ -0.707107\}\right\}$