Static Condensation, Settlements, Springs, Dependencies

Static Condensation

Static condensation is a technique that is possible in linear structural analysis to remove DOFs from the system of equations without locking them. The DOFs that are removed remain free to translate or rotate, but the value of the translation or rotation remains unknown after the system of equations in solved. Static condensation is particularly useful in the development of "super-elements." A super-element is one that is initially developed with "many" DOFs, some of which are removed by static condensation. Usually, only the DOFs at the boundary of the element are retained. To understand static condensation, consider the following sorted system of equations for an element or a structure

$$\begin{bmatrix} \mathbf{K}_{ii} & \mathbf{K}_{io} \\ \mathbf{K}_{oi} & \mathbf{K}_{oo} \end{bmatrix} \begin{Bmatrix} \mathbf{u}_i \\ \mathbf{u}_o \end{Bmatrix} = \begin{Bmatrix} \mathbf{F}_i \\ \mathbf{F}_o \end{Bmatrix} \tag{1}$$

where subscript i identifies the DOFs that are kept in, while subscript o identifies the DOFs that are to be tossed out. Next, write out the two sub-systems of equations

$$\begin{aligned} \mathbf{K}_{ii}\mathbf{u}_i + \mathbf{K}_{io}\mathbf{u}_o &= \mathbf{F}_i \\ \mathbf{K}_{oi}\mathbf{u}_i + \mathbf{K}_{oo}\mathbf{u}_o &= \mathbf{F}_o \end{aligned} \tag{2}$$

Solve for $\mathbf{u}_o$ in the second sub-system to obtain:

$$\mathbf{u}_o = \mathbf{K}_{oo}^{-1}\left(\mathbf{F}_o - \mathbf{K}_{oi}\mathbf{u}_i\right) \tag{3}$$

Substitution into the first sub-system yields:

$$\mathbf{K}_{ii}\mathbf{u}_i + \mathbf{K}_{io}\left(\mathbf{K}_{oo}^{-1}\left(\mathbf{F}_o - \mathbf{K}_{oi}\mathbf{u}_i\right)\right) = \mathbf{F}_i \tag{4}$$

Re-arrange to obtain the new system of equations in the unknowns $\mathbf{u}_i$:

$$\underbrace{\left(\mathbf{K}_{ii} - \mathbf{K}_{io}\mathbf{K}_{oo}^{-1}\mathbf{K}_{oi}\right)}_{\mathbf{K}}\ \mathbf{u}_i = \underbrace{\mathbf{F}_i - \mathbf{K}_{io}\mathbf{K}_{oo}^{-1}\mathbf{F}_o}_{\mathbf{F}} \tag{5}$$

where the new stiffness matrix, $\mathbf{K}$, and load vector, $\mathbf{F}$, for the super-element or super-structure without the $\mathbf{u}_o$ DOFs are identified.

Settlements and Imposed Deformations

DOFs that experience settlements and imposed displacements are not unknowns. Rather, it is the forces along those DOFs that are unknown. Consequently, the system of equations is reduced by the introduction of these effects. To understand how this is handled, consider the sorted system of equations for the structure

$$\begin{bmatrix} \mathbf{K}_{ii} & \mathbf{K}_{is} \\ \mathbf{K}_{si} & \mathbf{K}_{ss} \end{bmatrix} \begin{Bmatrix} \mathbf{u}_i \\ \mathbf{u}_s \end{Bmatrix} = \begin{Bmatrix} \mathbf{F}_i \\ \mathbf{F}_s \end{Bmatrix} \tag{6}$$

where subscript i identifies the free DOFs, while subscript s identifies the DOFs that are subject to settlement or imposed displacement or rotation. Next, write out the two sub-systems of equations

$$\begin{aligned} \mathbf{K}_{ii}\mathbf{u}_i + \mathbf{K}_{is}\mathbf{u}_s &= \mathbf{F}_i \\ \mathbf{K}_{si}\mathbf{u}_i + \mathbf{K}_{ss}\mathbf{u}_s &= \mathbf{F}_s \end{aligned} \tag{7}$$

The first of these two sub-systems is readily solved by moving the second term in the left-hand side over to the right-hand side:

$$\mathbf{K}_{ii}\mathbf{u}_i = \mathbf{F}_i - \mathbf{K}_{is}\mathbf{u}_s \tag{8}$$

because $\mathbf{u}_s$ is a vector of known displacements. Upon solving for $\mathbf{u}_i$, the second sub-system immediately provides the value of the forces $\mathbf{F}_s$ along the DOFs with imposed deformations.

Springs

Springs are concentrated stiffness values along selected DOFs. They typically represent flexible foundations, i.e., they are meant to be connected to ground. There are two ways to address concentrated springs. One approach is to introduce extra axial bars, i.e., truss elements that are attached to the structure at one end and to a fixed node at the other end. This is a straightforward approach, in which the bar stiffness EA/L is tuned to the desired spring stiffness. This approach also has the advantage that the force in the spring is read from the axial force in the truss member. Another approach is to introduce a special option to allow scalar stiffness values to be inserted into the diagonal of the structural stiffness matrix. This is perhaps simpler for the analyst because it eliminates the need to create an auxiliary node and truss element.

DOF Dependencies

A DOF dependency means that one DOF is equal to—or linearly dependent on—another DOF. One example is when the axial deformation of a horizontal frame element is neglected; then both horizontal end displacements are the same, i.e., dependent. Another example is an inclined roller support, in which case the vertical displacement is linearly related to the horizontal displacement. It is dangerous to address this problem by setting a member stiffness, e.g., axial stiffness, to some very high number. This may lead to poor conditioning of the stiffness matrix and, as a result, inaccuracies in the solution of the system of equations. A better approach is to introduce DOF dependencies by a special-purpose transformation matrix. For this purpose, consider the following relationship between DOFs in the All configuration:

$$\underbrace{\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \end{Bmatrix}}_{\mathbf{u}_a} = \underbrace{\begin{bmatrix} 0 & \alpha & 0 & \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \\ & \vdots & & \ddots \end{bmatrix}}_{\mathbf{T}_d} \underbrace{\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \end{Bmatrix}}_{\mathbf{u}_a} \tag{9}$$

where the transformation matrix $\mathbf{T}_d$ (d stands for dependency) states that DOF number one is equal to α times DOF number to, i.e.,

$$u_1 = \alpha \cdot u_2 \tag{10}$$

In other words, u_2 is the independent DOF and u_1 is the dependent DOF. As usual, the new stiffness matrix and load vector are obtained by

$$\mathbf{K}_a = \mathbf{T}_d^T \mathbf{K}_a \mathbf{T}_d, \qquad \mathbf{F}_a = \mathbf{T}_d^T \mathbf{F}_a \tag{11}$$

where the modified stiffness matrix has retained its dimension but has zeros in the row and column that correspond to the dependent DOF. Similarly, the modified load vector has a zero entry in the component that corresponds to the dependent DOF. Thus, the dependent DOF must be removed from the system of equations by the transformation from the All configuration to the Final configuration, i.e., by the $\mathbf{T}_{af}$ transformation matrix.