Vectors, Matrices, and Index Notation

Vectors and matrices are numbers or expressions organized in rows and columns. The numbers need not have the same unit. Consider the stiffness matrix for a cantilevered beam with two degrees of freedom:

$$\mathbf{K} = \begin{bmatrix} \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ -\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix} \tag{1}$$

This is a symmetric matrix with E=Young's modulus often measured in N/mm^2, I=moment of inertia often measured in mm^2, and L=is the length often measured in mm. Hence, the units of the entries of this matrix differ. An example of a matrix in which all entries are dimensionless is the correlation matrix

$$\mathbf{R} = \begin{bmatrix} 1 & 0.7 & 0.5 \\ 0.7 & 1 & 0.8 \\ 0.5 & 0.8 & 1 \end{bmatrix} \tag{2}$$

Similarly, vectors may have entries whose unit differ, such as the load vector

$$\mathbf{F} = \begin{Bmatrix} -\frac{qL}{2} \\ -\frac{qL^2}{12} \end{Bmatrix} \tag{3}$$

where q=distributed load often measured in N/mm. Conversely, the vector of standard normal random variables

$$\mathbf{y} = \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{Bmatrix} \tag{4}$$

is an example of a vector whose entries are dimensionless. Hence, it is possible to contemplate the "length" of the vector **y**:

$$||\mathbf{y}|| = \sqrt{y_1^2 + y_2^2 + y_3^2 + y_4^2} \tag{5}$$

while it is meaningless to talk about the length of **F**. Because of differing units, some computer programs, such as Mathcad, will complain if you use units when you work with stiffness matrices and load vectors.

Index Notation

Vectors and matrices, more generally called tensors, are perhaps best understood in index notation instead of the boldface notation used above. As an example, consider a generic system of linear equations, which is here written in five equivalent ways:

$$\mathbf{Ax}=\mathbf{b} \;\Leftrightarrow\; A_{ij}x_j=b_i \;\Leftrightarrow\; \sum_{j=1}^{3} A_{ij}x_j=b_i \;\Leftrightarrow\; \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \end{Bmatrix}$$

$$\Leftrightarrow \quad \begin{array}{ccc|c} & & & x_1 \\ & & & x_2 \\ & & & x_3 \\ \hline A_{11} & A_{12} & A_{13} & A_{11}x_1+A_{12}x_2+A_{13}x_3 \\ A_{21} & A_{22} & A_{23} & A_{21}x_1+A_{22}x_2+A_{23}x_3 \\ A_{31} & A_{32} & A_{33} & A_{31}x_1+A_{32}x_2+A_{33}x_3 \end{array} = \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \end{Bmatrix} \tag{6}$$

The last notation shows how you multiply a matrix and a vector by hand. The index notation $A_{ij}\,x_j=b_i$ is powerful when working with vector-matrix expressions. One reason is that the quantities can now be treated as regular scalars. Another reason is that index notation is similar to computer code. When looking at the product $A_{ij}\,x_j=b_i$, notice the similarity with the following C++ code:

```
for (int i=0; i<3; i++) {
    for (int j=0; j<3; j++) {
        b[i]=A[i][j]*x[j]  // Notice similarity with Aij xj=bi
    }
}
```

Caution must be applied when calculating that matrix-vector product on the computer, because the indices of vectors and matrices in some programming languages, such as C++ and Python, start at 0 instead of 1. An important rule of index notation is "summation over repeated indices:"

$$A_{ij}x_j = A_{i1}x_1 + A_{i2}x_2 + A_{i3}x_3 + \cdots \tag{7}$$

This is called "Einstein's summation convention" and states that summation should be taken over any index that appears twice in a term. In the case of $A_{ij}\,x_j$ that index is j. This makes j a "dummy index." Conversely, i is called a free index because it appears only once in each term. In the following examples of multiplication in vector and index notation, notice which are the free indices, and which are dummy indices. Also notice:

- The free indices reveal the dimension of the final result
- Unless there is a transpose the neighbouring indices are the same
- Transpose switches order of the indices of a matrix

- If there is a transpose, then neighbouring indices are no longer identical, but the range of the neighbouring indices must be the same
- The transpose of a vector does not affect the index notation; is an artifact of vectors being considered column vectors by default
- The ordering of the objects, i.e., vectors and matrices, in index notation is immaterial, while the order of the objects in vector notation is crucial

Free: *ik* $$\mathbf{AB} = A_{ij}B_{jk} \tag{8}$$

Free: *im* $$\mathbf{ABC} = A_{ij}B_{jk}C_{km} \tag{9}$$

Free: *i* $$\mathbf{Ax} = A_{ij}x_j \tag{10}$$

Free: *ik* $$\mathbf{A}^T\mathbf{B} = A_{ji}B_{jk} \tag{11}$$

Free: *i* $$\mathbf{A}^T\mathbf{x} = A_{ji}x_j \tag{12}$$

Free: *ik* $$(\mathbf{AB})^T = \left(A_{ij}B_{jk}\right)^T = A_{kj}B_{ji} = B_{ji}A_{kj} = \mathbf{B}^T\mathbf{A}^T \tag{13}$$

Free: none (scalar) $$\mathbf{x}^T\mathbf{Ax} = x_iA_{ij}x_j \tag{14}$$

Free: none (scalar) $$(\mathbf{Ax})^T(\mathbf{Ax}) = A_{ij}x_jA_{ik}x_k = x_jA_{ji}A_{ik}x_k = \mathbf{x}^T\mathbf{A}^T\mathbf{Ax} \tag{15}$$

Notice in particular the steps taken in Eq. (13). In the first equality it is decided that *i* and *k* are the free indices, hence the final result must have the indices *ik*, in that order. In the second equality the indices *ik* are flipped to *ki* because of the transpose. In the third equality, the order of *B* and *A* is flipped to achieve a final result that has indices *ik*. In the last equality it is recognized that both A and B must be transposed to make the dummy index *j* in *A* be a neighbour to the dummy index *j* in *B*. Also notice that the ordering of matrices and vectors in a product is important; consider for example

$$\mathbf{A} = \mathbf{BCD} \tag{16}$$

The product in Eq. (16) is not the same as the product **CDB** or other variations of the order. Notice also that the dimension of the matrices must match in the following sense: The number of columns in a preceding matrix must match the number of rows in the next matrix of a product. This is revealed by the indices in index notation of the same product:

$$A_{il} = B_{ij}C_{jk}D_{kl} \tag{17}$$

where *i* and *l* are free indices while *j* and *k* are dummy indices that are subjected to the Einstein summation convention.