Series Expansions

Series expansions of functions are useful for several purposes. It forms the basis for solution techniques for differential equations. Series expansion is also useful for simplifying nonlinear functions as a linear, quadratic, or sometimes higher-order polynomials. Yet another application is to evaluate functions with no closed form. There exist a host of special purpose series expansions. One generic technique is Taylor series expansions. Consider a nonlinear and differentiable function $f(x)$. Suppose it is unappealing to work with the exact function and that a series expansion about a particular point $x=x_0$ is contemplated. According to the Taylor series expansion the function is approximated by the following polynomial at points away from x_0:

$$f(x)=f(x_0)+f'(x_0)\cdot(x-x_0)+\frac{1}{2}\cdot f''(x_0)\cdot(x-a)^2+\frac{1}{3!}\cdot f'''(x_0)\cdot(x-x_0)^3+\cdots \quad (1)$$

where each prime as usual means one differentiation with respect to x. The accuracy of the approximation depends upon the number of terms that are included and the distance from x_0. The Taylor series for multi-variable functions reads, in vector notation:

$$f(\mathbf{x})=f(\mathbf{x}_0)+\nabla f(\mathbf{x}_0)^T(\mathbf{x}-\mathbf{x}_0)+\frac{1}{2}\cdot(\mathbf{x}-\mathbf{x}_0)^T H(\mathbf{x}_0)(\mathbf{x}-\mathbf{x}_0)+\cdots \quad (2)$$

where H is the Hessian matrix of second-derivatives, which is explicitly written in index notation:

$$f(x_i)=f+\frac{df}{dx_i}\cdot(x_i-x_{0,i})+\frac{1}{2}\cdot\frac{d^2f}{dx_i dx_j}\cdot(x_i-x_{0,i})\cdot(x_j-x_{0,j})+\cdots \quad (3)$$