Differential Equations

A differential equation is formulated in terms of an unknown function, which we wish to solve for. The function itself may or may not appear in the equation, but derivatives of the function certainly do; it is what makes it a differential equation. The equation that governs the beam displacement $w(x)$ in Euler-Bernoulli beam theory is one example:

$$\frac{d^4w}{dx^4} = \frac{q}{EI} \tag{1}$$

Sometimes the derivatives are written in shorthand notation, and here are some examples:

$$\frac{dw}{dx} \equiv w' \ , \quad \frac{d^2w}{dx^2} \equiv w'' \ , \quad \frac{dw}{dt} \equiv \dot{w} \ , \quad \frac{d^2w}{dt^2} \equiv \ddot{w} \ , \quad \frac{\partial h}{\partial x_i} \equiv \nabla h \ , \quad \frac{\partial \sigma_{xy}}{\partial x} \equiv \sigma_{12,1} \tag{2}$$

The second-to-last equality defines a "gradient vector" using the nabla symbol, while the last equality is common in index notation. The type of a differential equation is determined by its characteristic in each of the following five categories:

- **Ordinary Vs Partial:** The equation is ordinary if the derivatives of the unknown function are with respect to one parameter only; otherwise it is partial and the derivatives are written on the form $\partial w/\partial x$ instead of the ordinary form dw/dx.
- **Linear Vs Nonlinear:** If the unknown function and its derivative appear in linear form, e.g., without being raised to some power, then the equation is linear; otherwise it is nonlinear.
- **Homogeneous Vs Inhomogeneous:** If all the terms of the equation contain the unknown function or its derivative then the equation is homogeneous; otherwise it is inhomogeneous.
- **Constant Vs Variable Coefficients:** If the coefficients that multiply the unknown function and its derivatives contain any of the variables that the function depends on then the equation is said to have variable coefficients; otherwise it has constant coefficients.
- **Order:** The order of a differential equation is the highest order of derivative that appears in the equation.
- **Number of Terms:** The number of terms counts the number of instances of the derivative of the unknown function.

Figure 1 shows a flowchart to help select the appropriate solution approach for certain types of differential equations. The subsections below address each of these solution approaches.

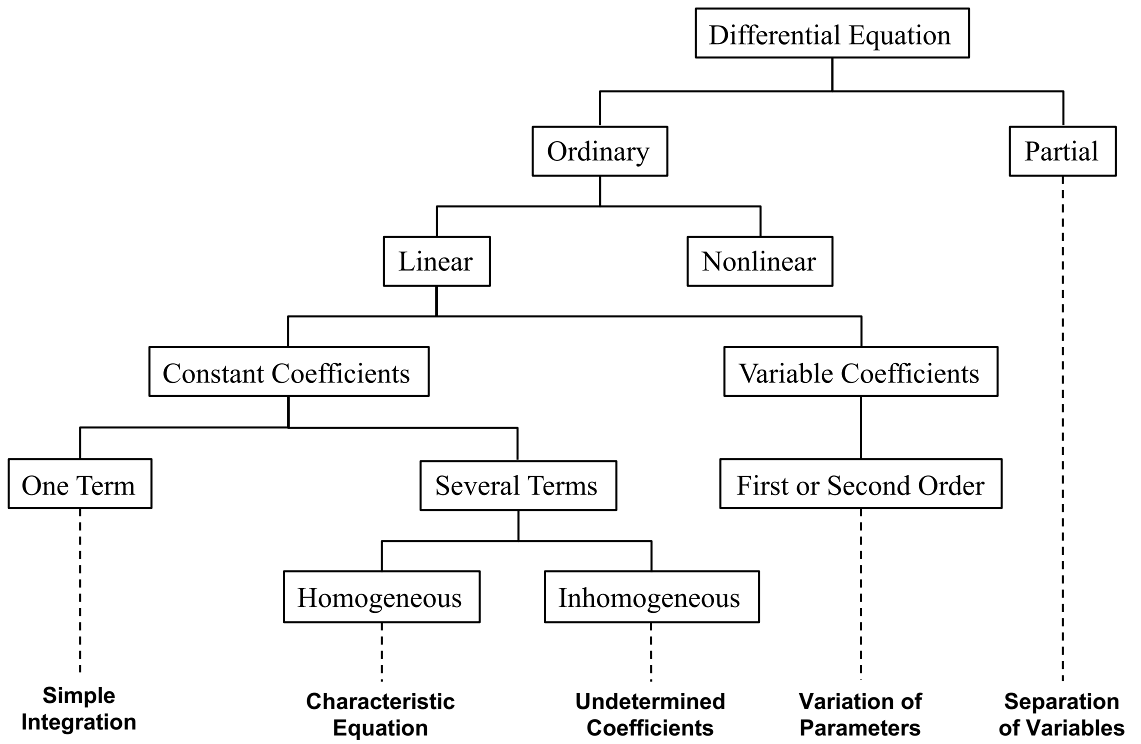

Figure 1: Identification chart for differential equations.

Simple Integration

If the differential equation has only one term that contains the unknow function then the solution approach is straightforward: integration. Consider the homogeneous equation

$$w'''' = 0 \tag{3}$$

Integrating four times yields the general solution:

$$w_h(x) = \frac{C_1}{6} \cdot x^3 + \frac{C_2}{2} \cdot x^2 + C_3 \cdot x + C_4 \tag{4}$$

where the subscript h is employed throughout this document to indicate the solution to a homogeneous equation and C_i are integration constants, to be determined from boundary conditions on w. The same approach is employed if the differential equation is inhomogeneous, such as this one:

$$w'''' = \frac{q}{EI} \tag{5}$$

where q and EI are here assumed to be constant, but could vary with x without change in the solution strategy. Integrating four times yields the solution

$$w(x) = \left(\frac{q}{12EI} x^4 \right) + \left(\frac{C_1}{6} \cdot x^3 + \frac{C_2}{2} \cdot x^2 + C_3 \cdot x + C_4 \right) = w_p(x) + w_h(x) \tag{6}$$

where the subscript p is employed to identify the "particular solution," i.e., that prompted by the inhomogeneous part.

Characteristic Equation

Generic examples of ordinary linear homogeneous differential equation with constant coefficients are

$$\begin{aligned} f' + a\cdot f &= 0 \\ f'' + a\cdot f' + b\cdot f &= 0 \\ f''' + a\cdot f'' + b\cdot f' + c\cdot f &= 0 \\ f'''' + a\cdot f''' + b\cdot f'' + c\cdot f' + d\cdot f &= 0 \end{aligned} \tag{7}$$

where a, b, c, and d are constants. Examples of such problems in structural analysis include the differential equation for a beam with axial force without lateral load:

$$w'''' + \frac{P}{EI}\cdot w'' = 0 \tag{8}$$

and the differential equation for combined St. Venant and warping torsion without distributed torque applied:

$$\phi'''' - \frac{GJ}{EC_w}\cdot \phi'' = 0 \tag{9}$$

The first step in the solution procedure is to establish the characteristic equation (Kreyszig 1988). It is defined as a polynomial in λ with order matching the order of the differential equation. For the problems in Eq. (7) the characteristic equations are

$$\begin{aligned} \lambda + a &= 0 \\ \lambda^2 + a\cdot\lambda + b &= 0 \\ \lambda^3 + a\cdot\lambda^2 + b\cdot\lambda + c &= 0 \\ \lambda^4 + a\cdot\lambda^3 + b\cdot\lambda^2 + c\cdot\lambda + d &= 0 \end{aligned} \tag{10}$$

The solution of the differential equation depends upon the values of the roots of the characteristic equation.

- If the roots are real and distinct then the solution is

$$f_h(x) = C_1\cdot e^{\lambda_1\cdot x} + C_2\cdot e^{\lambda_2\cdot x} + C_3\cdot e^{\lambda_3\cdot x} + \cdots \tag{11}$$

- If the roots are real but equal then the solution is

$$f_h(x) = \left(C_1 + C_2 x + C_3 x^2 + \cdots\right)\cdot e^{\lambda\cdot x} \tag{12}$$

- If the differential equation is second-order and the roots are complex conjugate numbers then the solution is

$$f_h(x) = e^{-\frac{a}{2}\cdot x}\left(C_1\cdot\cos(\omega\cdot x) + C_2\cdot\sin(\omega\cdot x)\right) \tag{13}$$

where

$$\omega = \sqrt{b - \frac{1}{4}a^2} \tag{14}$$

For higher-order differential equations, if the roots are complex and different, the solution is

$$f_h(x) = C_1 \cdot e^{\alpha_1 \cdot x} \cdot \cos(\omega_1 \cdot x) + C_2 \cdot e^{\alpha_1 \cdot x} \cdot \sin(\omega_1 \cdot x) + \text{ possibly other pairs} \tag{15}$$

where

$$\lambda_i = \alpha_i + i \cdot \omega_i \tag{16}$$

Conversely, for higher-order differential equations, if the roots are complex and equal, the solution is

$$\begin{aligned} f_h(x) &= C_1 \cdot e^{\alpha \cdot x} \cdot \cos(\omega \cdot x) + C_2 \cdot e^{\alpha \cdot x} \cdot \sin(\omega \cdot x) \\ &+ C_3 \cdot x \cdot e^{\alpha \cdot x} \cdot \cos(\omega \cdot x) + C_4 \cdot x \cdot e^{\alpha \cdot x} \cdot \sin(\omega \cdot x) \\ &+ C_5 \cdot x^2 \cdot e^{\alpha \cdot x} \cdot \cos(\omega \cdot x) + C_6 \cdot x^2 \cdot e^{\alpha \cdot x} \cdot \sin(\omega \cdot x) \\ &+ \cdots \end{aligned} \tag{17}$$

Undetermined Coefficients

When ordinary linear differential equation with constant coefficients have an inhomogeneous part then the total solution is written as the sum of the homogenous solution, obtained as described above, and the particular solution:

$$f(x) = f_h(x) + f_p(x) \tag{18}$$

Examples include Eq. (8) with applied load:

$$w'''' + \frac{P}{EI} \cdot w'' = \frac{q_z}{EI} \tag{19}$$

and Eq. (9) with distributed torque:

$$\phi'''' - \frac{GJ}{EC_w} \cdot \phi'' = \frac{m_x}{EC_w} \tag{20}$$

The method of undetermined works if the right-hand-side, i.e. the load, is an exponential function, a polynomial, a cosine or sine function, or a sum of such functions (Kreyszig 1988). If the right-hand-side of the differential equation contains the term $k \cdot e^{\gamma x}$ then

$$f_p(x) = C \cdot e^{\gamma x} \tag{21}$$

If the right-hand-side contains the term $k \cdot x^n$, n=0,1,2,… then

$$f_p(x) = C_n \cdot x^n + C_{n-1} \cdot x^{n-1} + C_1 \cdot x + C_0 \tag{22}$$

If the right-hand-side contains $k \cdot \cos(\omega x)$ or $k \cdot \sin(\omega x)$ then

$$f_p(x) = C_A \cdot \cos(\omega x) + C_B \cdot \sin(\omega x) \tag{23}$$

The unknown constants C are determined by substituting the solution into the differential equation. If the right-hand-side contains several functions then $f_p(x)$ is the sum of the corresponding solutions give above. If a term in the particular solution appears in the homogeneous solution then that term should be multiplied by x. It should be multiplied by x^2 if that solution corresponds to a double root of the characteristic equation (Kreyszig 1988).

Variation of Parameters

The method of variation of parameters is a more general but also more complicated way of finding solutions to inhomogeneous differential equations. This is more rare in structural analysis, but Section 2.16 of Kreyszig's textbooks is a good starting point (Kreyszig 1988) if needed.

Separation of Variables

A partial differential equation that appears in structural analysis is the "one-dimensional wave equation:"

$$\frac{\partial^2 w(x,t)}{\partial t^2} = c^2 \cdot \frac{\partial^2 w(x,t)}{\partial x^2} \tag{24}$$

The solution approach is to first separate the variables, then to solve the two separate differential equations, and finally satisfy the boundary conditions. The starting point is to write the solution as (Kreyszig 1988)

$$w(x,t) = F(x) \cdot G(t) \tag{25}$$

Differentiation of Eq. (25) with respect to t yields

$$\ddot{w} = F \cdot \ddot{G} \tag{26}$$

Differentiation of Eq. (25) with respect to x yields

$$w'' = F'' \cdot G \tag{27}$$

Substitution of Eqs. (26) and (27) into Eq. (24) and dividing through by c^2FG yields

$$\frac{\ddot{G}}{c^2 G} = \frac{F''}{F} \tag{28}$$

Because the right-hand-side depends only on t while the right-hand-side depends only on x both sides must be constant. That constant is here denoted $-K$ and as a result the following two ordinary linear homogeneous differential equations appear:

$$\ddot{G} + K \cdot c^2 \cdot G = 0 \tag{29}$$

$$F'' + K \cdot F = 0 \tag{30}$$

The final step depends upon the boundary conditions. One solution is found in the document on oscillating strings and beams.