Common Functions

A function describes the relationship between some input parameter(s) and some output parameter(s). The simplest function is abstractly written $y=f(x)$, which states that the value of the dependent variable y varies with the value of the independent variable x according to the function f. Figure 1 and this document provides an overview of some common functions in mathematics.

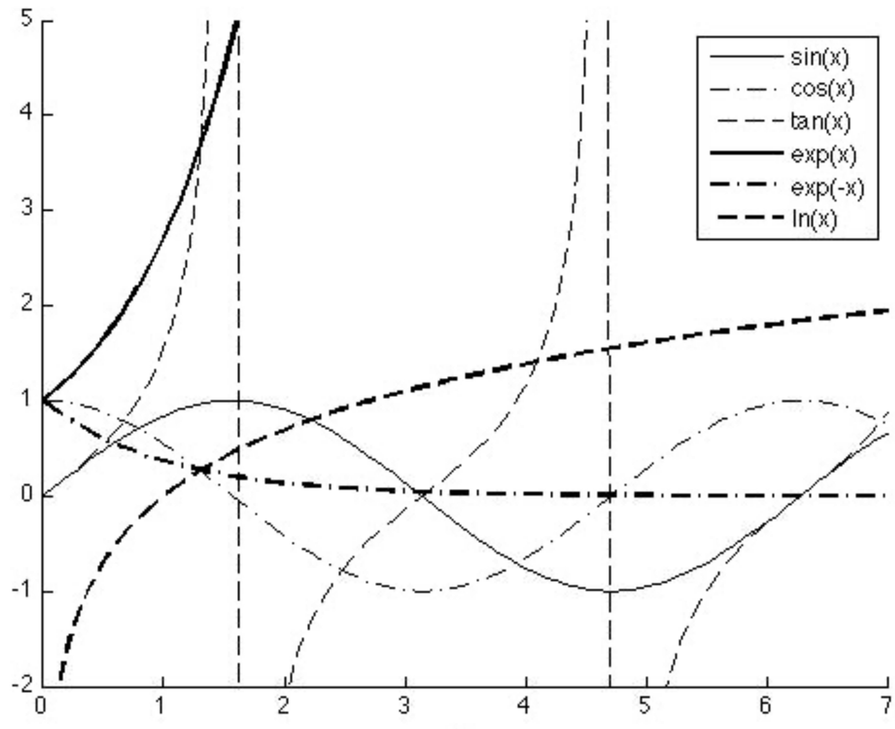

Figure 1: A plot of a few mathematical functions.

These functions stem from the field of geometry that deals with triangles. To repeat from geometry; the sine and cosine functions are the essential ones. The other functions, i.e., tangent, cotangent, secant, and cosecant can be related to those two. The value of the sine and cosine functions for some key angles is:

θ (degrees)	θ (radians)	$\cos(\theta)$	$\sin(\theta)$
0°	0	$\frac{1}{2}\sqrt{4} = 1$	$\frac{1}{2}\sqrt{0} = 0$
30°	$\pi/6$	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}\sqrt{1} = \frac{1}{2}$
45°	$\pi/4$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{2}$
60°	$\pi/3$	$\frac{1}{2}\sqrt{1} = \frac{1}{2}$	$\frac{1}{2}\sqrt{3}$
90°	$\pi/2$	$\frac{1}{2}\sqrt{0} = 0$	$\frac{1}{2}\sqrt{4} = 1$

The sine and cosine functions have a period of 2π radians (180° degrees). Waves of other periods are obtained by using the argument

$$\frac{2\pi}{T}\cdot x \tag{1}$$

where T is the desired period. The sine and cosine functions cross zero every π radians (90° degrees), hence

$$\cos\left(\left(n+\tfrac{1}{2}\right)\cdot\pi\right)=\sin\left(n\cdot\pi\right)=0 \quad \text{for} \quad n\in\mathbb{Z} \tag{2}$$

The arcsine(x), arcos(x), and arctan(x) functions return the angle in the equations x=sin(θ), x=cos(θ), and x=tan(θ). In general, an exponential function has a base and a power, such as

$$f(x)=a^x \tag{3}$$

where a is the base and x is the power. Of particular significance is the exponential function with base e, where

$$e=\lim_{t\to 0}\left(\left(1+t\right)^{1/t}\right)\approx 2.7183 \tag{4}$$

With this base, the exponential function, denoted e^x or exp(x), equals its derivative. Regardless of base, some rules are:

$$a^0=1 \tag{5}$$

$$a^{-x}=\frac{1}{a^x} \tag{6}$$

$$a^{p/q}=\sqrt[q]{x^p}=\left(\sqrt[q]{x}\right)^p \tag{7}$$

$$a^p\cdot a^q=a^{p+q} \tag{8}$$

$$\frac{a^p}{a^q}=a^{p-q} \tag{9}$$

$$\left(a^p\right)^q=a^{p\cdot q} \tag{10}$$

$$\left(a\cdot b\right)^p=a^p\cdot b^p \tag{11}$$

$$\left(\frac{a}{b}\right)^p=\frac{a^p}{b^p} \tag{12}$$

The natural logarithm ln(x) of a number x is the power that e must be raised to, to obtain x:

$$e^{ln(x)}=x \tag{13}$$

Logarithmic functions with other base numbers than e are available, most prominently base-10. Notation wise, *ln* implies base-e and *log* usually implies base-10. However, *log* is sometimes used with base-e. Whichever base, the logarithm function obeys these rules:

$$\ln(a \cdot b) = \ln(a) + \ln(b) \tag{14}$$

$$\ln\left(\frac{a}{b}\right) = \ln(a) - \ln(b) \tag{15}$$

$$\ln(a^x) = x \cdot \ln(a) \tag{16}$$

The rule in Eq. (16) is particularly useful when taking the logarithm on both sides of an equation to solve for an unknown that appears in a power. Hyperbolic trigonometric functions appear in the solution of some differential equations. Their definitions are:

$$\sinh(x) = \frac{1}{2}\left(e^x - e^{-x}\right) \tag{17}$$

$$\cosh(x) = \frac{1}{2}\left(e^x + e^{-x}\right) \tag{18}$$

The definition of the hyperbolic version of tangent, cotangent, secant, and cosecant follows that of ordinary trigonometry. Figure 2 visualizes the hyperbolic trigonometric functions.

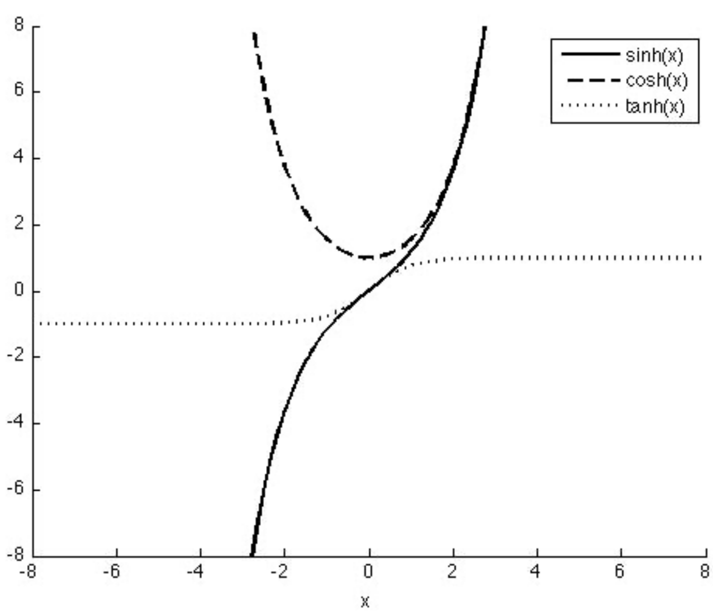

Figure 2: The hyperbolic trigonometric functions.

Dirac's delta function is defined so that

$$\delta(x - x_0) = \begin{cases} \infty & \text{for } x = x_0 \\ 0 & \text{elsewhere} \end{cases} \tag{19}$$

and

$$\int_{-\infty}^{\infty} \delta(x - x_0)\,dx = 1 \tag{20}$$

Heaviside's step function is the integral of Dirac's delta function:

$$H(x - x_0) = \int_{-\infty}^{x} \delta(t)\,dt = \begin{cases} 0 & \text{for } x < x_0 \\ 1 & \text{for } x \geq x_0 \end{cases} \tag{21}$$

Conversely, Dirac's delta function is the derivative of Heaviside's step function:

$$\delta(x - x_0) = \frac{d}{dx} H(x - x_0) \tag{22}$$

Consider a generic function, $f(x)$, and its second-derivative with respect to x, namely $f''(x)$. As illustrated in Figure 3, the function $f(x)$ is called convex when $f''(x)>0$ and concave when $f''(x)<0$.

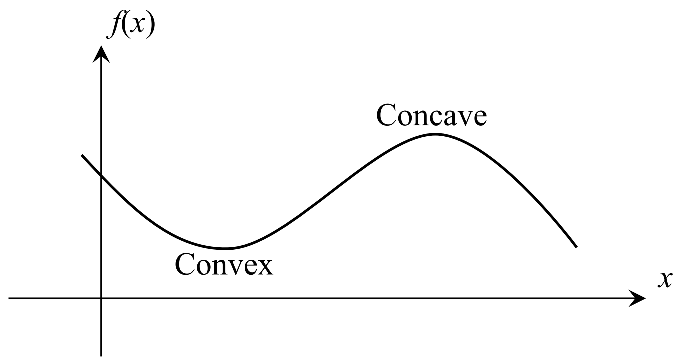

Figure 3: Convexity and concavity of functions.