Combinatorics

When dealing with a collection of n objects, this question sometimes arises: How many unique combinations are possible when k objects are drawn from the collection? The answer is provided by the binomial coefficient:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n \cdot (n-1) \cdot (n-2) \cdots (n-k+1)}{k \cdot (k-1) \cdot (k-2) \cdots (1)} \quad \text{for } k \leq n \tag{1}$$

One special case is when k=n, i.e., the case where all objects are drawn. Then, intuitively, there is only one combination, namely the collection of all objects. Appropriately, Eq. (1) then evaluates to unity. Another special case is when k=1, i.e., when only one object is drawn. Intuitively, there is then as many unique “combinations” as there are objects. Accordingly, Eq. (1) then evaluates to n.

A somewhat related problem is to obtain the total number of combinations for all possible k. The answer is:

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n \tag{2}$$

A simpler problem is the determination of the number of joint states of a set of discrete variables. Let m denote the number of variables, and let each variable be identified by the index i, where i=1,…,m. Each variable has s_i number of states. As an example, consider a problem with three variables with the following states:

- Variable 1 has three states: A, B, C
- Variable 2 has two states: A, B
- Variable 3 has four states: A, B, C, D

For this example, the total number of joint states is $s_1 s_2 s_3$=(3)(2)(4)=24. In general, it is

$$\prod_{i=1}^{m} s_i \tag{3}$$

which simplifies to

$$k^m \tag{4}$$

when all variables have k possible states.