Calculus

Differentiation and integration are the two central topics in calculus. The former produces the rate-of-change of a function with respect to one or more variables, while the latter is essentially a summation technique. Differentiation and integration are reciprocal: integration of a differentiated function yields the original function, and vice versa. Consider one function of one variable, $f(x)$. The first-order derivative of f with respect to x can be written in two ways:

$$\frac{df(x)}{dx} \equiv f'(x) \tag{1}$$

Similarly, the second-order derivative is written

$$\frac{d^2 f(x)}{dx^2} \equiv f''(x) \tag{2}$$

Derivatives of a multi-variable function with respect to one or more variables are called partial derivatives. The following demonstrate the notation for a few examples:

$$\frac{\partial f(x,y)}{\partial x} \quad , \quad \frac{\partial^2 f(x,y)}{\partial x^2} \quad , \quad \frac{\partial^2 f(x,y)}{\partial x \partial y} \tag{3}$$

The derivative of a vector, i.e., gradients, is often written with the nabla symbol:

$$\frac{df(\mathbf{x})}{d\mathbf{x}} \equiv \frac{df(x_i)}{dx_i} \equiv \nabla f(\mathbf{x}) \tag{4}$$

In index notation the derivative of vectors, matrices, and higher-order tensors are often written with a comma-notation:

$$\frac{\partial \sigma_{ij}}{\partial x_i} \equiv \sigma_{ij,i} \tag{5}$$

Product rule of differentiation:

$$\frac{d}{dx}\big(f(x)\cdot g(x)\big) = \frac{df(x)}{dx}\cdot g(x) + f(x)\cdot\frac{dg(x)}{dx} \tag{6}$$

Chain rule of differentiation:

$$\frac{d}{dx} f\big(g(x)\big) = \frac{df(g)}{dg}\cdot\frac{dg(x)}{dx} \tag{7}$$

Differentiation of polynomials:

$$\frac{d}{dx} x^n = n \cdot x^{n-1} \tag{8}$$

Differentiation of some trigonometric functions:

$$\begin{aligned}\frac{d}{dx}\cos(x) &= -\sin(x)\\ \frac{d}{dx}\sin(x) &= \cos(x)\end{aligned} \tag{9}$$

Differentiation of exponential functions:

$$\begin{aligned}\frac{d}{dx}e^{x} &= e^{x}\\ \frac{d}{dx}e^{f(x)} &= e^{f(x)}\cdot\frac{df(x)}{dx}\\ \frac{d}{dx}a^{x} &= a^{x}\cdot\ln(a)\\ \frac{d}{dx}a^{c\cdot x} &= a^{c\cdot x}\cdot\ln(a)\cdot c\end{aligned} \tag{10}$$

Differentiation of logarithmic functions:

$$\begin{aligned}\frac{d}{dx}\ln(x) &= \frac{1}{x}\\ \frac{d}{dx}\ln\big(f(x)\big) &= \frac{1}{f(x)}\cdot\frac{df(x)}{dx}\end{aligned} \tag{11}$$

The derivative of the Heaviside function yields the Dirac delta function, as described in the document on functions. Integration, denoted by the symbol $\int$, is essentially a summation of infinitely many small contributions. There are definite and indefinite integrals. In the former the integration boundaries are stated, while the latter requires subsequent specification of integration boundaries to obtain a specific value. The evaluation of a definite integral is carried out in two steps. First, the indefinite integral is evaluated:

$$\int f(x)\,dx = I(x) \tag{12}$$

Next, the integration boundaries are applied:

$$\int_a^b f(x)\,dx = I(b) - I(a) \equiv \big[I(x)\big]_a^b \tag{13}$$

Indefinite integration by parts:

$$\int f'(x)\cdot g(x)\,dx = f(x)\cdot g(x) - \int f(x)\cdot g'(x)\,dx \tag{14}$$

Definite integration by parts:

$$\int_a^b f'(x)\cdot g(x)\,dx = \big[f(x)\cdot g(x)\big]_a^b - \int_a^b f(x)\cdot g'(x)\,dx \tag{15}$$

Integration by substitution for indefinite integrals:

$$\int f\big(g(t)\big)\cdot g'(t)\,dt = \int f(x)\,dx \tag{16}$$

Integration by substitution for definite integrals:

$$\int_a^b f\big(g(t)\big)\cdot g'(t)\,dt = \int_{g(a)}^{g(b)} f(x)\,dx \tag{17}$$

Indefinite integration always generates an integration-constant. It is denoted by C in the following. Indefinite integration of polynomials:

$$\int x^n\,dx = \frac{1}{n+1}\cdot x^{n+1} + C \tag{18}$$

Indefinite integration of trigonometric functions:

$$\begin{aligned}\int \cos(x)\,dx &= \sin(x) + C\\ \int \sin(x)\,dx &= -\cos(x) + C\end{aligned} \tag{19}$$

Indefinite integration of exponential functions:

$$\begin{aligned}\int e^x\,dx &= e^x + C\\ \int a^x\,dx &= \frac{a^x}{\ln(a)} + C\end{aligned} \tag{20}$$

Indefinite integration of logarithmic functions:

$$\begin{aligned}\int \ln(x)\,dx &= x\cdot\ln(x) - x + C\\ \int \log_a(x)\,dx &= x\cdot\log_a(x) - \frac{x}{\ln(a)} + C\end{aligned} \tag{21}$$

The indefinite integral of the Heaviside function is

$$\int H(x-x_0)\,dx = (x-x_0)\cdot H(x-x_0) \tag{22}$$

If the integral of that result is sought, which it is for instance when Dirac's delta function is utilized to model point load on beams, it is computed with integration by parts:

$$\int (x-x_0)\cdot H(x-x_0)\,dx = (x-x_0)\cdot(x-x_0)\cdot H(x-x_0) - \int (x-x_0)\cdot H(x-x_0)\,dx \tag{23}$$

Solving for the integral yields

$$\int (x-x_0)\cdot H(x-x_0)\,dx = \frac{1}{2}\cdot(x-x_0)^2\cdot H(x-x_0) \tag{24}$$

In words, when integrating a function multiplied by the Heaviside function, the Heaviside function is pulled out of the integral, and its argument is treated as a single variable in the subsequent integrations.