Decision Criteria

The motivation for this document is the need to make decisions under uncertainty. Classical examples include a farmer deciding which type of crop to plant, under uncertain future weather conditions. Another example is an engineer deciding the dimensions of structural members, under uncertain capacity and loads. That engineer may not feel uncertain if he/she relies on prescriptive codes that give those numbers, but behind the codes are decisions under uncertainty. The theory of decision making under uncertainty finds applications in many other endeavours, including finance and the everyday life of humans. Documentation of the sometimes amazing and sometimes flawed decision-making by humans is found in many newspapers and books (Gladwell 2005; Kahneman 2011). It is also interesting that modern neuroscience is providing new insight into the complex processes, perhaps algorithms, that are behind even the simplest of human decisions. However, two comments related to the purpose of this document are made:

1. It is an objective here to *recommend decision criteria* rather than to mimic how humans actually make decisions. The latter often includes the modelling of irrational behaviour, which does not seem productive for engineering decisions.
2. It is an objective to include the consideration of *major one-off decisions*, such as designing a lifeline bridge or betting the farm. Such decisions are made infrequently and the impacts if something goes wrong are potentially devastating.

How Do Humans Make Decisions?

The answer to that question is not always pleasant. Humans often make decisions irrationally, based on emotions, unaware of faulty instincts, and overestimating the probability of rare events (Kahneman 2011; Kahneman and Tversky 1979; Tversky and Kahneman 1992). On the other hand, it is perhaps wrong to expect humans to act like machines, always optimizing based on rational models. In fact, philosophers such as Baruch Spinoza (1632-1677) and Arne Næss (1912-2009) have argued that the role of emotions is crucial and often undervalued when we talk about "rational" decisions.

How do engineers, developers, and society make structural design decisions? Usually based on formulas in building codes and material standards. Those formulas are handed down by code committees and contain prescribed safety coefficients that are applied to seemingly deterministic load and resistances values. The safety coefficients are usually calibrated to some target reliability index, β, typically above β=3, which implies a failure probability below $10^{-2.87}$, but rarely above β=4, which implies a $10^{-4.5}$ failure probability. What those probabilities imply may be unclear but they are meant to match currently accepted practice. Experienced engineers often add important cost-benefit considerations to the design process, but rarely using optimization algorithms and explicit models. When researchers do that they find risk averse or even risk seeking designs compared with the rational optimum (Cha and Ellingwood 2012; Mahsuli and Haukaas 2018).

But what is the "rationally optimal design?" It is the design that maximizes the total expected utility, including all costs and benefits, of the facility over its lifetime from material extraction to demolition or deconstruction (von Neumann and Morgenstern 1944). For example, a hospital should be safer than a one-car garage because the failure costs are different. The mantra of expected utility theory is behind much of the material posted on this website but it is not universally accepted in structural design, for two reasons:

1. Expected utility theory requires models for *many costs and benefits*. This is a big effort that involves probabilistic modelling. Environmental impacts, cost of injuries, and other *intangible costs* must be quantified. This is a rational approach but still awkward for many humans.
2. All concerns must be *translated into a unified measure of utility*, i.e. euros or dollars. The weight of a particular concern is often debatable. In fact, the *utility of a facility may vary by stakeholder*; a developer may be concerned primarily with profit, while members of the general society may be more concerned about economic growth or environmental damage.

Among researchers who pursue the rational optimization approach there is little debate about Item 1. Any optimization analysis requires one or more objectives and Item 1 addresses the modelling of those objectives. Item 2 is sometimes circumvented by the use of multi-objective optimization algorithms, avoiding a summation of concerns into one utility. However, this is not a rational solution; at some point the different concerns must be weighed to arrive at a unique design decision. To understand this, imagine you have to choose between two jobs, one with $70,000 salary and 8 weeks of vacation per year, the other with $110,000 salary and 2 weeks of vacation. Unless you leave the decision to a multi-objective optimization algorithm you must quantify how much vacation time is worth to you.

Expected Cost: Pascal & de Fermat 1654

In 1654 Blaise Pascal and Pierre de Fermat invented the theory of probability in an exchange of letters that is now legendary (Ore 1960). That correspondence also formalized a decision criterion that was instinctively understood by gamblers long before: over many games it is the *expected* (average) gain that matters. Loss in one game does not matter if there is a net gain over many games. In fact, when inventing the concept of probability, Pascal and de Fermat addressed two gambling-related problems. The problems were posed by Antoine Gombaud chevalier de Meré, a respectable member of society who had nonetheless made certain observations related to gambling (Ore 1960). Addressing one of de Meré's problems, Pascal and de Fermat devised mathematical techniques to split the stakes fairly between two gamblers whose multi-game contest is disrupted. The solution is to give each gambler the amount that corresponds to their *expected* gain, if the full series of games had been played out. Notice that the frequency notion of probability underpins this decision criterion: only by considering the average, i.e., the expectation over many completions of the disrupted contest is it possible to understand the fraction of the stakes given to each player. This yields a powerful rationale for expected cost as a decision criterion: by making the decision that minimizes the expected cost the gains are maximized over many decisions. But is this criterion sensible for one-off decisions with potentially major negative implications if the expectation is not realized?

Expected Utility: Daniel Bernoulli 1738

Expected utility theory implies an extension of expected cost as a decision criterion. It does not enjoy the advantage that the gains will be maximized over many decisions. However, it addresses an important concern with expected cost: the decision-maker may be devastated by negative outcomes before he or she gets to enjoy the gains over many decisions. For example, the farmer may go bankrupt the first or second season because of bad weather, before accumulating money from good seasons. This issue is referred to as risk aversion and the first solution was formulated by Daniel Bernoulli (1738). He did it in response to a paradox posed by his cousin Nicolaus Bernoulli, who formulated the problem in a letter to Pierre Raymond de Montmort in 1713. Nicolaus described a situation where the use of expected cost as a decision criterion gives an unreasonable result. Suppose you are offered a game that involves repeatedly tossing a fair coin. Every time the coin is tossed there are two possible outcomes: heads and tails. Before starting the game, you must pay a fixed amount to participate and there is a pot of money, the "stakes," which you win once the first tail appears. Importantly, the stakes start at \$1 and double at every coin toss until the first tail appears and you win. As a result, the amount you will win is a discrete random variable whose expected (mean) value is what you should be willing to pay to enter the game. The possible wins are \$1, \$2, \$4, \$8, etc. This continues to infinity, although it is unlikely that the game will go on for very long without a tail appearing. To compute the expected value of what you will win, the probability associated with each possible win is required. With a fair coin, the probability of \$1 is 0.5. The probability of \$2, i.e., the probability of heads in both the first and the second throw of the coin, is, assuming independence, $0.5^2=0.25$. In short, the expected value of the win is

$$E[w]=\frac{1}{2}\cdot\$1+\frac{1}{4}\cdot\$2+\frac{1}{8}\cdot\$4+\frac{1}{16}\cdot\$8+\cdots=\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\cdots=\infty \tag{1}$$

In words, the expected cost criterion suggests that you should be willing to pay an infinite amount of money to participate in this game. That is likely unreasonable to you because there is a 50% chance that you will leave the table with a win of only \$1. This is known as the St. Petersburg paradox, named after the city where Nicolaus and also Daniel Bernoulli were professors for a period of time. The pioneering solution by Daniel Bernoulli is to introduce a utility function that expresses the "diminishing marginal utility" of money. According to Bernoulli: "There is no doubt that a gain of one thousand ducats is more significant to the pauper than to a rich man although both gain the same amount." Bernoulli suggested to use *expected utility instead of expected cost* as the decision criterion. He suggested as utility the natural logarithm of the monetary value. This relationship between money and utility is shown in Figure 1 as what is now known as a "utility function" or "Bernoulli function." The expected utility of the game is

$$E[u]=\frac{1}{2}\cdot\ln(\$1)+\frac{1}{4}\cdot\ln(\$2)+\frac{1}{8}\cdot\ln(\$4)+\frac{1}{16}\cdot\ln(\$8)+\cdots=0.693 \tag{2}$$

with corresponding dollar value $e^{0.693}$=\$2. For most people that is a more reasonable fee to enter the game.

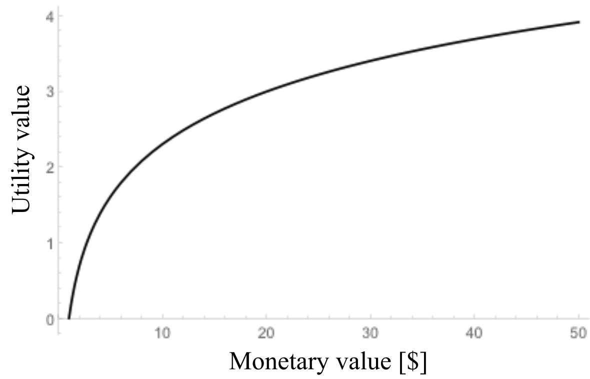

Figure 1: Daniel Bernoulli's logarithmic utility function.

Expected Utility: von Neumann & Morgenstern 1944

After Daniel Bernoulli's publication of a utility function in 1738 it took 200 years until the next blossom in that field. The start of the second spring is marked by the publication of a book by the economist Oskar Morgenstern and the mathematician John von Neumann. In that book the axioms and mathematical foundation of utility theory were established in the context of economics (von Neumann and Morgenstern 1944). The treatment by von Neumann and Morgenstern is quite mathematical, but several subsequent works elucidate the approach (Benjamin and Cornell 1970; Jordaan 2005; Raïffa and Schlaifer 2000; Wald 1950). A key issue in utility theory is to model the decision maker's risk aversion in terms of utility functions, such as Bernoulli's function shown in Figure 1. To establish a decision maker's utility function it is helpful to think of the "basic reference lottery tickets" described in Section 4.4.1 in the book by Jordaan (2005). This approach follows the idea in Section 3.3.2 of von Neumann and Morgenstern (1944) to let a decision maker express a preference between an outcome A and a lottery with possible outcomes B and C. In his book, Jordaan (2005) also cites Raiffa (1968) to be the first to formalize the following lottery question:

> *Suppose you face a decision problem in which the worst possible outcome is denoted W and the best possible outcome is denoted B. At a specific dollar amount, D, whose value is between W and B you are asked the following question:*
>
> *Suppose you are offered D in cash with no strings attached. At which probability, u, would you be willing to NOT take D but rather enter into a lottery with probability u of winning B and probability 1–u of getting the worst outcome, W.*
>
> *The value of u is the utility associated with the dollar amount D.*

Depending on the magnitude of the potential loss *W*, many people will require a high value of *u*, i.e., a high probability of winning *B*, before entering into the lottery. As a result, the utility function is usually concave, such as the one shown in blue in Figure 2. Conversely, people who have a propensity for gambling will cherish the excitement of entering into any lottery with the possibility of winning *W*. Hence, the gambler's answers will form a convex curve like the one shown in red in Figure 2. Importantly, the utility functions are subjective; a specific amount of money may have different utility for different people. A linear function, shown in black in Figure 2, implies that that money and utility are interchangeable, meaning that the decision is based on expected cost. That is the sensible approach for a decision-maker with a deep pocket facing many decisions: using expected cost as decision criterion maximises the benefits over time.

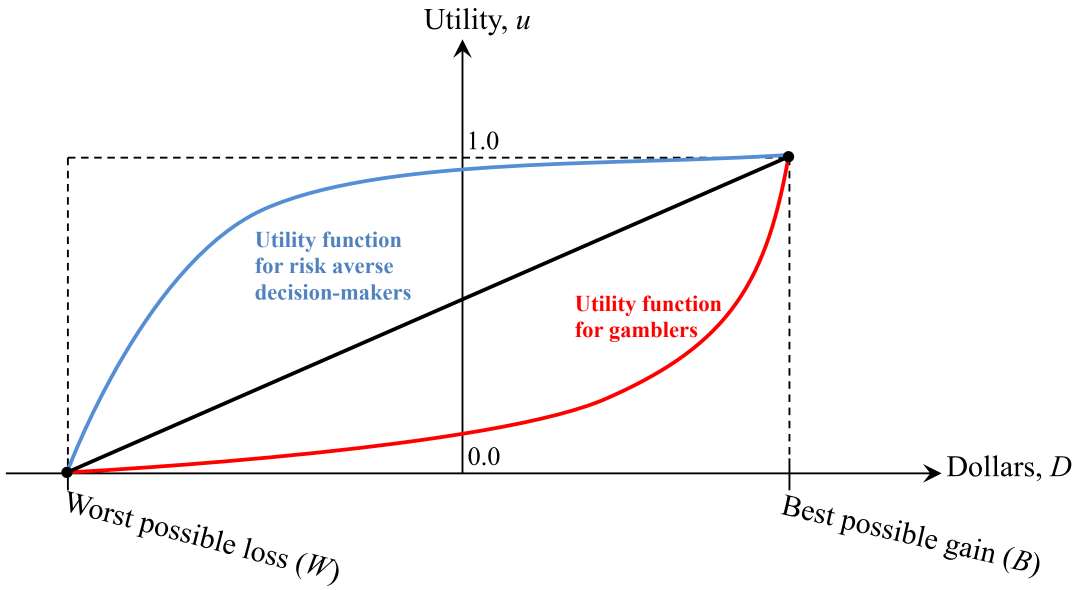

Figure 2: Utility functions.

Critiques of Expected Utility Theory

Compared with expected cost as decision criterion, expected utility introduces risk aversion to help decision makers who may go under if the worst possible loss is realized. The downside is that the choies are no longer optimal over many decisions. Large resources may be unnecessarily committed to ensuring a safe outcome. That is the fundamental trade-off when applying expected utility theory, but several other issues have been raised:

- Tversky and Kahneman carried out pioneering studies to determine the actual behaviour of people facing decisions under uncertainty (Kahneman 2011). They found that people overestimate the probability of rare but devastating outcomes. They also found that the perceived utility of gains and losses are often related to some intermediate reference value, rather than W and B mentioned above. Tversky and Kahneman developed “prospect theory” to remedy those shortcomings of utility theory, and another document on this website describes some aspects of that theory. However, the modelling of people’s sometimes irrational behaviour is not further pursued here, where the objective is to recommend rational engineering decisions.
- Another critique of utility theory is the difficulties associated with extracting the decision maker’s “true” utility function (Friedman et al. 2014). An infinite space of curves is available, but there is limited evidence that any one utility function is the objectively true one for any given decision maker.

- Utility theory necessitates the mapping of various concerns onto one common axis with units of dollars or some other generic measure of utility. To some, this is a disadvantage because real-world decisions have multiple and often contradictory objectives. However, two comments may alleviate this concern. First, for those who say that real-world decisions under risk can only be made through a process of comprehensive and complex stakeholder engagements, with lengthy meetings where all involved parties get to express their thoughts and concern, then technical decision criteria will invariably play a secondary role. Second, even multi-objective decision techniques require the weighing of different concern, conceptually equivalent to the mapping of different concerns into one common axis with units of dollars or some other generic measure of utility.

Problem Types

Different problem types appear within the framework of expected utility theory. In the following it is assumed, without loss of generality, that cost is the measure of utility. Hence, the key tasks are to evaluate and minimize the expected cost. Which techniques to employ for this purpose depend on 1) characteristics of the intervening design variables, 2) characteristics of the cost, and 3) whether the option of purchasing perfect or imperfect data is available. Table 1 suggests one way of categorizing the problems, together with the name of respective solution techniques, which are described in the subsequent sections.

Table 1: Types of decision problems under uncertainty.

	Discrete cost	Continuous cost	
		With limit-states	No limit-states
Discrete design variables	**Decision trees**	-	-
Continuous design variables	-	**RBDO**	**Using the mean**

Decision Trees

With discrete design variables and discrete cost the methodology presented in Raiffa and Schlaifer (2000) and Benjamin and Cornell (Benjamin and Cornell 1970) is applicable. The decision alternatives are called actions and denoted A_i. Each action is associated with a cost, $c(A_i)$. The possible outcomes are denoted θ_j and each outcome is associated with a cost, $c(\theta_j|A_i)$, and a probability of occurrence, $P(\theta_j)$. Before drawing the decision tree it is necessary to:

1. Enumerate the decision alternatives, A_i
2. Compute the cost of each decision alternative, $c(A_i)$
3. Enumerate the possible outcomes, θ_j
4. Assess the cost of each outcome, $c(\theta_j|A_i)$; often organized in a “payoff table” that often includes $c(A_i)$
5. Compute the probability of each outcome, $P(\theta_j)$

The objective of the decision tree analysis is to identify the action with lowest expected cost. Or conversely, if the cost values are translated into utility values the objective is to identify the action with highest expected utility. The setup of a decision tree to determine the expected cost of each action is shown in Figure 3. The starting point is the left-most decision fork, which is drawn as a rectangle. Each action branch ends at a chance fork, which is drawn as a circle. As illustrated in Figure 3, the expected cost of each action branch is

$$E\left[c \middle| A_i\right] = c(A_i) + \sum_{j=1}^{J} c(\theta_j | A_i) \cdot P(\theta_j) \tag{3}$$

where J is the total number of outcomes. The action with the lowest expected cost, or equivalently the one with highest expected utility, is the optimal decision. However, two other decision strategies exist: 1) Minimize the maximum cost, which may be selected by a risk-averse decision maker in one-off situations; 2) Maximize the possible benefit, which may be selected by a gambling-inclined decision maker.

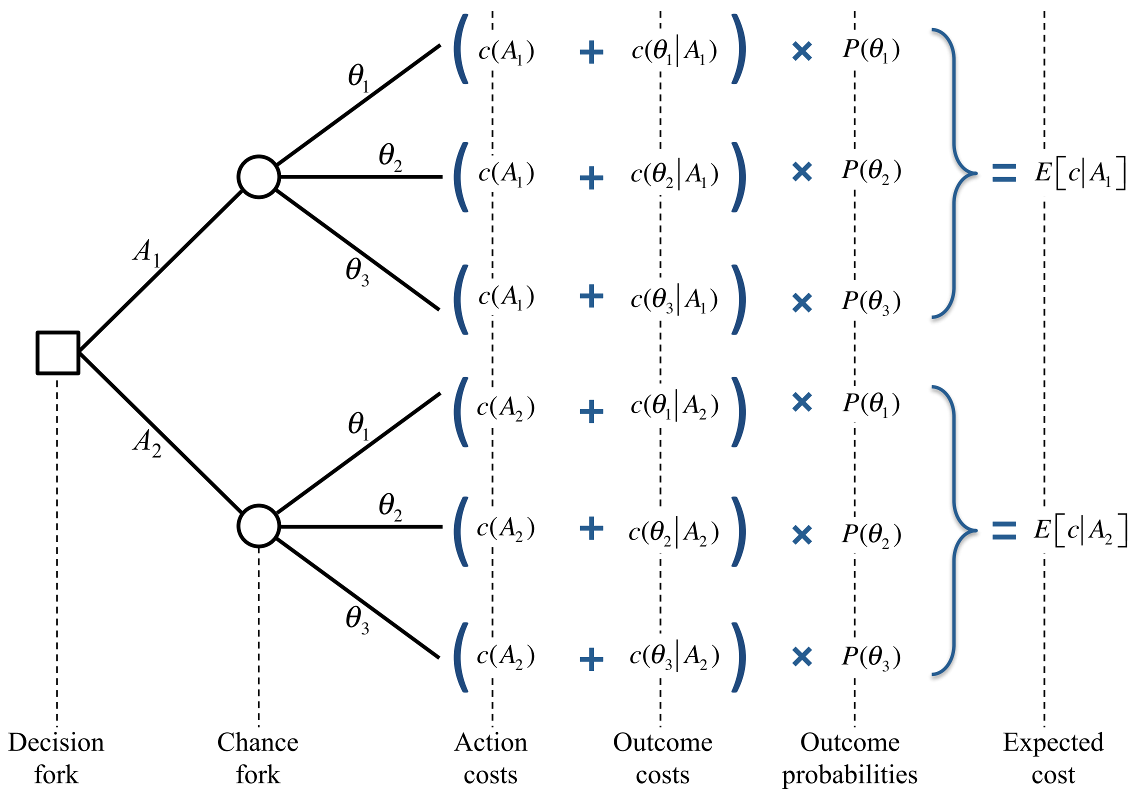

Figure 3: Decision tree.

Terminal Analysis

Terminal analysis extends the decision tree in circumstances when new information becomes available. The information is not conclusive; otherwise the decision would be easy. Rather, the information is associated with some uncertainty. The typical example is test data obtained with an imperfect device, from which the probability of various outcomes is provided as "sample likelihoods," i.e., in the conditional form

$$P\left[I_k \middle| \theta_j\right] \tag{4}$$

where I_k is the indicator value from the test device and θ_j is the real state. For each j, i.e., for each possible real state, the following condition must be satisfied:

$$\sum_{k=1}^{K} P\left[I_k \middle| \theta_j\right] = 1 \tag{5}$$

where K is the number of possible test indicator values. The key step in terminal analysis is to use the test probabilities in Eq. (4) to update the outcome probabilities in the decision tree. I.e., the objective is to compute the probabilities P[$\theta_j|I_k$]. Notice that the outcome probabilities are dependent on the indicator value from the test device; for each indicator value there will be a unique decision tree, as shown in Figure 4. The new indicator-dependent outcome probabilities are computed by Bayes' rule:

$$P\left[\theta_j \middle| I_k\right] = \frac{P\left[I_k \middle| \theta_j\right]}{P\left[I_k\right]} P\left[\theta_j\right] \tag{6}$$

For each indicator branch in Figure 4 a basic decision tree is drawn, with outcome probabilities from Eq. (6). That is, for each indicator value there will be an optimal decision that should be made if that indicator value is observed.

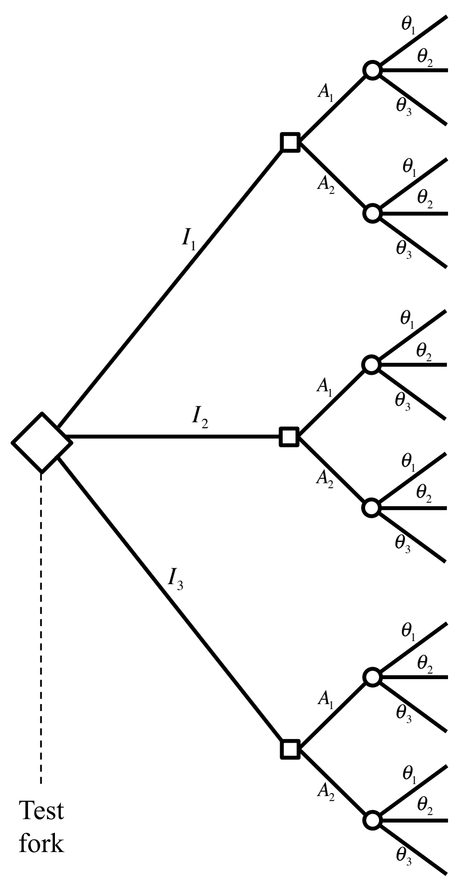

Figure 4: Decision tree for terminal analysis.

Pre-posterior Analysis

Now consider the problem of deciding whether to purchase test information. Two situations arise; the information may be perfect or associated with uncertainty. Consider first the case of perfect test information. The expected cost in a situation where the test removes all uncertainty about the outcome, albeit excluding the cost of the test itself, is

$$E\left[c_{\text{test}}\right]=\sum_{j=1}^{J}\left[\left(c(A_{oj})+c(\theta_j\middle|A_{oj})\right)\cdot P(\theta_j)\right] \quad (7)$$

where J is the number of possible outcomes (remember, the test device is perfect) and A_{oj} is the obvious decision once θ_j is the known outcome. Eq. (7) states that the expected cost in a perfect-test situation is the sum of the probability of each possible outcome multiplied by the certain cost of the action that is obvious once that outcome is known. To determine whether it is cost-effective to purchase the test, the expected cost in Eq. (7) is compared with the expected cost of the optimal decision in the classical decision tree in Figure 3. The expected cost in Eq. (7) is lower and the discrepancy is the saving associated with having perfect test information. It is expected to be cost-effective to purchase the test if the purchase-cost does not exceed this saving. Another type of pre-posterior analysis arises when deciding whether to pay for imperfect information. Then, the probability for the possible test results (indicator values) are computed by the rule of total probability:

$$P\left[I_k\right]=\sum_{i=1}^{J}P\left[I_k|\theta_j\right]\cdot P\left[\theta_j\right] \quad (8)$$

Each of these probabilities, P[I_k], is multiplied by the expected cost of the optimal decision that follows the test result I_k, which is available from a terminal analysis shown in Figure 4. The complete pre-posterior decision tree for this situation is shown in Figure 5. Summation of these products yields the expected cost in the situation where we have help from an imperfect test device. This value is compared with the expected cost of the optimal decision in the classical decision tree in Figure 3. The discrepancy is the saving associated with performing and imperfect test. It is expected to be cost-effective to purchase the test if the purchase-cost does not exceed this saving.

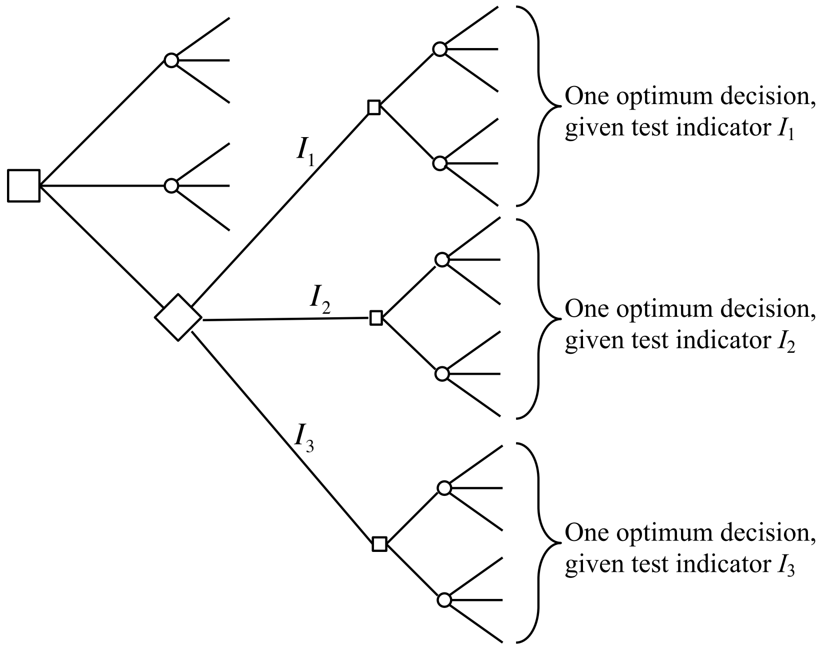

Figure 5: Decision tree for pre-posterior analysis with the option of selecting imperfect test.

Reliability-based Design Optimization (RBDO)

Reliability is defined as unity minus the failure probability, limit-state function(s) defining failure. Hence, the label reliability-based implies that one or more failure probabilities are involved in the problem formulation. For these problems it is possible to imagine discrete design variables and even discrete costs, but the concepts are here explained with continuous variables and continuous costs. The typical formulation of expected cost is

$$\mathrm{E}[c] = c_0(\mathbf{x}) + c_f \cdot p_f(\mathbf{x}) \tag{9}$$

where c_0 is the expected cost of construction, $\mathbf{x}$ are the design variables, c_f is the expected cost of failure, and p_f is the probability of failure. Figure 6 schematically identifies the optimal design for the case of one design variable based on the minimization of the expected cost. As a conceptual illustration, the figure indicates an increasing construction cost due to increasing design variable value, and a decreasing expected cost of failure due to decreasing failure probability as the design variable value increases. The sum of both, which is the total expected cost, is a curve with a unique minimum. This point represents the optimum design and thereby the optimum failure probability.

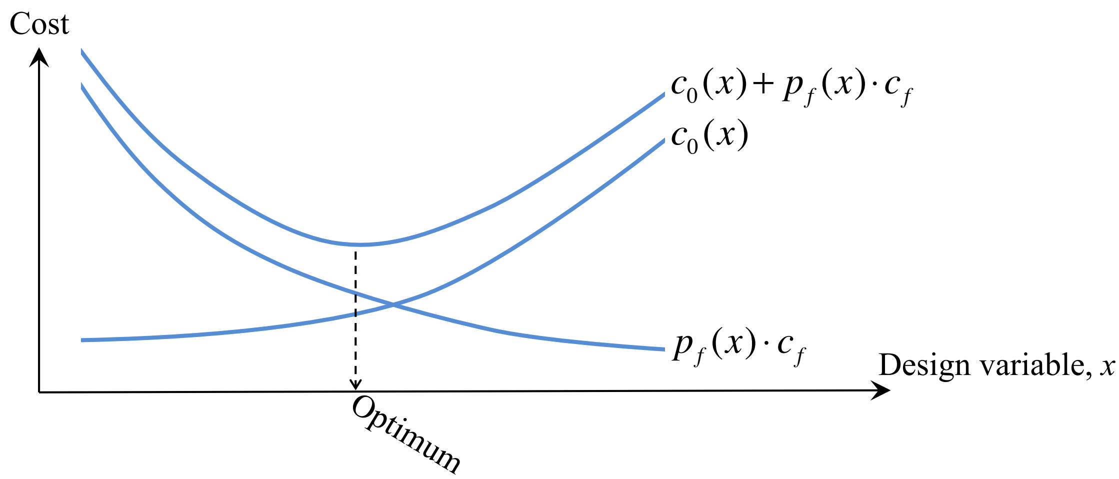

Figure 6: Schematic illustration of reliability-based design optimization.

When constraints on the failure probability and the design variables are included, the reliability-based design optimization problem reads

$$\mathbf{x}^* = \arg\min\left\{c_0(\mathbf{x}) + c_f \cdot p_f(\mathbf{x}) \;\middle|\; \left(p_f(\mathbf{x}) - p_0\right) \leq 0, \quad \mathbf{f}(\mathbf{x}) \leq 0\right\} \tag{10}$$

where the asterisk identifies the optimal design. The first constraint expresses the requirement that the failure probability is less than the threshold p_0. However, one may argue that for the reliability-based optimization problem in Eq. (10) it is unnecessary to include a probability constraint. This is because the cost of failure is already included in the objective function. If the cost of failure is properly modelled, including the potential for human injury and other intangible costs, then the optimization analysis will provide the "best" target safety, without the need for explicit constraints. The optimal failure probability will then come out high for a garage, which has a low cost of failure, while it will come out low for a hospital.

Using the Mean

Consider a problem where the outcome is a continuous random variable. Performance-based earthquake engineering is one example; there, are key result is the probability distribution for cost, including repair of potential future damage (Cornell and Krawinkler 2000; Haukaas 2008; Yang et al. 2009). These problems can be formulated without limit-state functions, and conceptually the problem is now simpler than the one shown in Eq. (10) and Figure 6. The total cost has many contributions and is essentially a continuous random variable, whose possible distributions are schematically shown in Figure 7. According to the expected cost mantra, the optimal design is that which minimizes the mean of that random variable. The left-hand side of Figure 7 shows two cost distributions with identical expected cost but differing variance. Design 1 can be thought of as a well-tested concept, while Design 2 is a more experimental design with larger uncertainty in its performance. Using expected cost as a decision criterion there is *no preference* between Design 1 and Design 2, although Design 2 is associated with a higher probability of sustaining a realization with large cost. The right-hand side of below shows a third design option with higher expected benefit but also higher variance. Paradoxically to some, the expected cost mantra favours Design 3 although the probability of seeing high cost realizations is larger than with Design 1. This highlights a key aspect in expected cost/utility decisions: the strategy is optimal over many decisions as long as the decision maker can handle intermediate intermittent realizations of large costs.

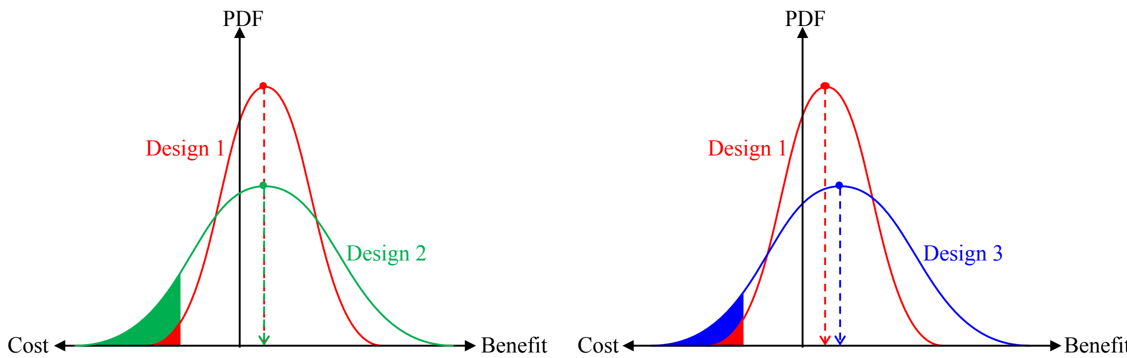

Figure 7: Schematic probability distributions of total cost.

Realizations of the random variable whose probability distributions are sketched in Figure 7 are calculated with many models that use many random variables and many design variables (Mahsuli and Haukaas 2013a; b; Yang et al. 2009). This implies that no standard distribution type is available, and that it is analytically and computationally prohibitive to establish the full and exact distribution. Thankfully, to carry out expected cost optimization, only the mean value is needed and this can be estimated in several ways:

- Monte Carlo sampling
- First-order second-moment Taylor approximation (gradient available)
- Second-order second-moment Taylor approximation (gradient available if response-Hessian is available)
- First-order reliability method at a grid of cost thresholds (gradient available)
- Response expansion methods
- Dimension reduction techniques

Prospect Theory

Prospect theory was developed as a descriptive model of how humans make decisions, based data collected in questionnaires (Kahneman 2011; Kahneman and Tversky 1979). A simplistic way of viewing prospect theory is as an extension of expected utility theory. In the same way as utility theory adds the concept of nonlinear utility to expected cost theory, prospect theory adds moving parts to utility theory. However, prospect theory has fundamental differences with utility theory, and perhaps a better perspective is that prospect theory is a psychology-based description of how actual decisions are made, rather than a foundation for rational engineering decisions. Regardless, important lessons about loss aversion can be learned from applying prospect theory in an engineering context.

The key concepts in prospect theory are a reference point and aversion to losses from that reference point. For most decision-makers, the reference point changes from decision to decision. For example, experiencing a success shifts the reference point for the next decision. Prospect theory postulates that decisions are made based on changes in wealth relative to the reference point. While losses and gains in utility theory are measured by going from point to point on the utility curve, prospect theory allows different utilities to be assigned to a loss and a win, even when the loss and the win are of equal nominal value. In other words, prospect theory focuses on changes in utility rather than particular states of utility. Figure 3 epitomizes prospect theory and shows how the same amount lost generates a stronger negative experience than the positive experience of winning the same amount.

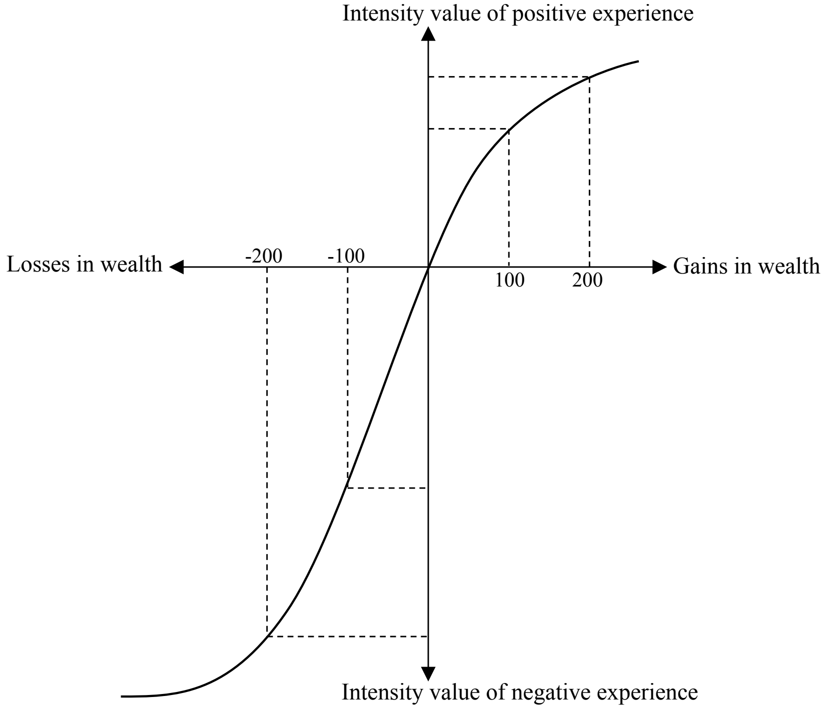

Figure 8: Value function in prospect theory (Kahneman 2011).

In utility theory, the expected utility is

$$E[u] = p_1 \cdot u_1 + p_2 \cdot u_2 + \cdots + p_n \cdot u_n \tag{11}$$

where p_i are probabilities, u_i are outcome utilities, and $p_1+p_2+\ldots+p_n=1$. In prospect theory, the probabilities are replaced by decision weights $\pi(p)$ and the utilities are replaced by values $\nu(u)$ defined relative to a reference point (Kahneman and Tversky 1979). Figure 3 shows a schematic example of a value function and Figure 9 shows a schematic example of a weighing function. The decision weights, π, need not add to unity, and the prospect value

$$V = \pi(p_1)\cdot \nu(u_1) + \pi(p_2)\cdot \nu(u_2) + \cdots + \pi(p_n)\cdot \nu(u_n) \tag{12}$$

can be interpreted as a relaxation of the expectation principle of utility theory.

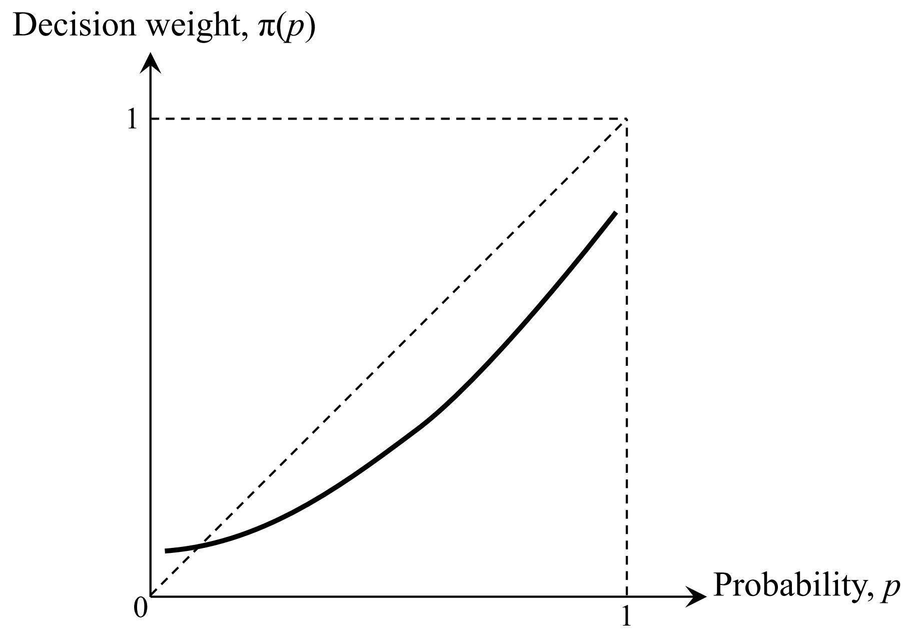

Figure 9: Weighting function (solid line) in prospect theory (Kahneman and Tversky 1979).

An extension of the theory outlined above is called cumulative prospect theory (Tversky and Kahneman 1992). This decision-making model has been employed to seismic risk problems (Goda and Hong 2008) and to general infrastructure exposed to low-probability high-consequence events (Cha and Ellingwood 2012).

The Life Quality Index

The LQI (life quality index) is an index that measures incremental changes in the health and safety of a population. If money spent on an infrastructure project yields an increase in the LQI then the resource allocation is meaningful, otherwise not. Although the LQI is philosophically interesting and has received significant attention in the structural reliability community, the translation into effective decision-making for individual structures remains an interesting challenge. Another challenge is that the benefit to society of human actions can arguably be measured in different ways. For example, reduction in the cumulative pain and suffering that individuals in a population experiences may be another indicator that should drive societal research allocation (Kahneman 2011). Thus, in the following derivations one should be mindful of the complexities associated with estimating what constitutes improvement in life quality. In fact, other documents on this website focus on direct consequence modelling and minimization of total cost or some other direct measure of utility.

Several research groups have contributed to the development of the LQI. The group centered at the University of Waterloo in Canada introduced the index in a seminal book (Nathwani et al. 1997) that was later revised and expanded (Nathwani et al. 2009). The same researchers discussed a broader set of economic risk acceptance criteria (Lind 2002) and provided a derivation of the LQI economic principles (Pandey et al. 2006). The Canadian group also introduced the SWTP concept, i.e., the societal willingness to pay

(Pandey and Nathwani 2004). Research led by Professor Rackwitz at the Technical University of Munich applied the LQI in reliability-based optimal design (Rackwitz 2002). This group also addressed the influence of discounting and other factors (Rackwitz 2006), and applied the LQI to aging infrastructure (Rackwitz and Joanni 2009) and seismic risk mitigation (Sánchez-Silva and Rackwitz 2004). Research at the Technical University of Danmark led by Professor Ditlevsen discussed the LQI in the concext of decision-making under uncertainty (Ditlevsen 2003) and provided revisions of the original formulation (Ditlevsen 2004) that became a subject of debate (Rackwitz 2005). The Danish group also introduced the “life quality time allocation index” (Ditlevsen and Friis-Hansen 2005) and they provided a discussion of the Canadian research group’s derivation of the LQI from economic principles (Ditlevsen and Friis-Hansen 2008).

The two key ingredients of the LQI are: 1) the life expectancy at birth, e, and 2) the real gross domestic product per person, g. The intensity of life quality is quantified by a function $f(g)$, and the duration of the good life is introduced by a function $h(t)$, where t=(1-w)e is the time spent enjoying life, where w is the fraction of life spent in production of g rather than life enjoyment. The compound measure of life quality is

$$L = f(g)\cdot h(t) \tag{1}$$

The change in L, measured by the differential is, using the product rule of differentiation:

$$dL = \left(\frac{df}{dg}\cdot dg\right)\cdot h + f\cdot\left(\frac{dh}{dt}\cdot dt\right) \tag{2}$$

Normalizing by L yields:

$$\begin{aligned}\frac{dL}{L} &= \frac{1}{f\cdot h}\cdot\left(\frac{df}{dg}\cdot dg\right)\cdot h + \frac{1}{f\cdot h}\cdot f\cdot\left(\frac{dh}{dt}\cdot dt\right)\\ &= \frac{1}{f}\cdot\frac{df}{dg}\cdot dg + \frac{dh}{dt}\cdot\frac{1}{h}\cdot dt\end{aligned} \tag{3}$$

Multiplying the first term by g/g and the second term by t/t, defining the constants k_1 and k_2, and realizing that dt/t equals de/e yields:

$$\begin{aligned}\frac{dL}{L} &= \frac{g}{f}\frac{df}{dg}\cdot\frac{dg}{g} + \frac{dh}{dt}\cdot\frac{dt}{t}\cdot\frac{t}{h}\\ &= \underbrace{\left(\frac{g}{f}\frac{df}{dg}\right)}_{k_1}\cdot\frac{dg}{g} + \underbrace{\left(\frac{dh}{dt}\cdot\frac{t}{h}\right)}_{k_2}\cdot\frac{dt}{t}\\ &= k_1\cdot\frac{dg}{g} + k_2\cdot\frac{de}{e}\end{aligned} \tag{4}$$

The constants k_1 and k_2 quantify the relative impact of increasing g, i.e., improving the intensity of life enjoyment, and the life expectancy, e, i.e., the duration of life enjoyment. Assuming that the ratio k_1/k_2 remains constant implies that k_1 and k_2 are constants, which leads to the two equations:

$$
\begin{aligned}
k_1 &= \frac{g}{f}\frac{df}{dg} \\
k_2 &= \frac{t}{h}\cdot\frac{dh}{dt}
\end{aligned}
\tag{5}
$$

These differential equations can be written in the following form:

$$
\begin{aligned}
&\frac{df}{dg} - \frac{k_1}{g}\cdot f = 0 \\
&\frac{dh}{dt} - \frac{k_2}{t}\cdot h = 0
\end{aligned}
\tag{6}
$$

These are ordinary linear homogeneous differential equations with variable coefficients. The general solution to such equations is described in a math document on this website, and the specific solutions are:

$$
\begin{aligned}
f(g) &= C_1 \cdot e^{-\ln(g)\cdot k_1} = C_1 \cdot g^{-k_1} \\
h(t) &= C_2 \cdot e^{-\ln(t)\cdot k_2} = C_2 \cdot t^{-k_2}
\end{aligned}
\tag{7}
$$

In the product $f(g)h(t)$ the constants C_1 and C_2 are irrelevant, and renaming k_1 and k_2 yields (Nathwani et al. 1997):

$$
\begin{aligned}
f(g) &= g^q \\
h(t) &= t^s
\end{aligned}
\tag{8}
$$

so that the LQI in Eq. (1) takes the form:

$$
\begin{aligned}
L &= g^q \cdot t^s \\
&= g^q \cdot \left((1-w)\cdot e\right)^s
\end{aligned}
\tag{9}
$$

To obtain expressions for q and s it is postulated that the gross domestic product is proportional to the time spent in production. I.e., g=we so that

$$
L = \left(w\cdot e\right)^q \cdot \left((1-w)\cdot e\right)^s \tag{10}
$$

Assuming that L is in practice maximized by people balancing their time producing g and their leisure time, i.e., assuming that people live at the optimal fraction w yields:

$$
\frac{dL}{dw} = 0 \tag{11}
$$

The solution to Eq. (11) is

$$
q = \left(\frac{w}{1-w}\right)\cdot s \tag{12}
$$

Setting q=w and s=1-w satisfies Eq. (12). Substitution of these values into Eq. (9) yields:

$$
L = g^w \cdot e^{1-w} \cdot (1-w)^{1-w} \tag{13}
$$

The last factor is practically constant, leading to the following expression for the LQI:

$$L = g^w \cdot e^{1-w} \tag{14}$$

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