Size Effect Models

Weibull Theory

This theory is based on a "weakest link" approach to material strength, and essentially simplifies the problem to the calibration of two parameters m and k. To understand the theory, consider a structural member that consists of n small elements. If any of the small elements fail, then the member fails. Denote the strength of each small element by τ and assume that the probability distribution of this strength, i.e., $F(\tau)$ is known. Then, the survival probability of each element, $p_{s,n}$ is

$$p_{s,n} = 1 - F(\tau) \tag{1}$$

If the small elements are statistically independent then the survival probability for the entire member is

$$p_s = \left(1 - F(\tau)\right)^n \tag{2}$$

For large n, series expansion yields

$$p_s = e^{-n \cdot F(\tau)} \tag{3}$$

Consequently, the failure probability of the member, for large n, is

$$p_f = 1 - e^{-n \cdot F(\tau)} \tag{4}$$

According to Weibull, the CDF for the strength of each small element is

$$F(\tau) = \left(\frac{\tau}{m}\right)^k \tag{5}$$

for $\tau>0$, where $k>0$ is called the shape parameter and $m>0$ is called the scale parameter. Eq. (5) is intended as a convenient approximation of the lower tail of the unknown actual probability distribution of the strength. Substitution of Eq. (5) into (4) yields

$$p_f = 1 - e^{-n \cdot \left(\frac{\tau}{m}\right)^k} \tag{6}$$

If the structural member is divided into infinitely many infinitesimally small elements then the failure probability reads:

$$p_f = 1 - e^{-\int_V \left(\frac{\tau}{m}\right)^k \mathrm{d}V} \tag{7}$$

Under the assumption that the structural member has unit volume and uniform stress, the failure probability becomes:

$$p_f = 1 - e^{-\left(\frac{\tau^*}{m}\right)^k} \tag{8}$$

where τ^* is the strength of the unit reference volume. Eq. (8) is the standard Weibull distribution. Suppose this reference strength is known. The strength of another structural member non-unit volume and/or non-uniform stress distribution is obtained by equating its failure probability with the unit-volume reference member:

$$e^{\int_V \left(\frac{\tau}{m}\right)^k \mathrm{d}V} = e^{\left(\frac{\tau^*}{m}\right)^k} \quad \Rightarrow \quad \int_V \tau^k \,\mathrm{d}V = \left(\tau^*\right)^k \tag{9}$$

Hence, a design check can be of the form

$$\tau^* \geq \left(\int_V \tau^k \,\mathrm{d}V \right)^{\frac{1}{k}} \tag{10}$$

or equivalently of the form

$$\frac{\tau^*}{\left(\int_V \tau^k \,\mathrm{d}V \right)^{\frac{1}{k}}} \geq 1.0 \tag{11}$$

To arrive at the O86 code equation for shear capacity of large glulam beams, multiply Eq. (11) by the constant W_f on both sides

$$W_f \cdot \frac{\tau^*}{\left(\int_V \tau^k \,\mathrm{d}V \right)^{\frac{1}{k}}} \geq W_f \tag{12}$$

The shear stress distribution over a rectangular cross-section is

$$\tau(x,y,z) = \left(\frac{3}{2 \cdot A} - \frac{6 \cdot z^2}{A \cdot d^2} \right) \cdot V(x) \tag{13}$$

Hence, the volume integral in Eq. (12) reads

$$\begin{aligned} \int_V \tau^k \,\mathrm{d}V &= b \cdot \int_{-\frac{d}{2}}^{\frac{d}{2}} \left(\frac{3}{2 \cdot A} - \frac{6 \cdot z^2}{A \cdot d^2} \right)^k \mathrm{d}z \int_0^L V(x)^k \,\mathrm{d}x \\ &\approx 3.3 \cdot \frac{A}{A^k} \cdot \int_0^L V(x)^k \,\mathrm{d}x \qquad (14) \\ &= 3.3 \cdot \frac{Z}{A^k} \cdot \frac{1}{L} \cdot \int_0^L V(x)^k \,\mathrm{d}x \end{aligned}$$

Substitution into Eq. (12) yields

$$\tau^* \cdot A \cdot 3.3^{-\frac{1}{k}} \cdot W_f \cdot \left(\frac{L}{\int_0^L V(x)^k \,\mathrm{d}x} \right)^{\frac{1}{k}} \cdot Z^{-\frac{1}{k}} \geq W_f \tag{15}$$

In contrast, the O86 code formula is

$$\begin{aligned} V_r &= \phi \cdot F_v \cdot 0.48 \cdot A_g \cdot K_N \cdot C_v \cdot Z^{-0.18} \\ &= 0.9 \cdot \left(f_v \cdot K_D \cdot K_H \cdot K_{Sv} \cdot K_T \right) \cdot 0.48 \cdot A_g \cdot (1.0) \cdot C_v \cdot Z^{-0.18} \\ &\geq \mathrm{W}_f \end{aligned} \tag{16}$$

where the factor C_v is

$$C_v = 1.825 \cdot W_f \cdot \left(\frac{L}{\sum G} \right)^{0.2} = 1.825 \cdot W_f \cdot \left(\frac{L}{6 \cdot \int_0^L |V|^5 \,\mathrm{d}x} \right)^{0.2} \tag{17}$$

where G is substituted by the integral of the shear force diagram because the code specifies

$$G = l_a \cdot \left(V_A^5 + V_B^5 + 4 \cdot V_C^5 \right) \tag{18}$$

which is Simpson's integration rule without the factor 1/6. It is also noted that W_f is the total load, Z is the beam volume, and A_g is the cross-sectional area. In summary, the O86 code equation is

$$\phi \cdot F_v \cdot A \cdot 0.61 \cdot W_f \cdot \left(\frac{L}{\int_0^L |V|^5 \,\mathrm{d}x} \right)^{0.2} \cdot Z^{-0.18} \geq W_f \tag{19}$$

Compared this with the theoretically derived Eq. (15).

Possible Simpler Implementation

$$\int_V \tau^k \,\mathrm{d}V = \left(\tau^* \right)^k \tag{20}$$

$$\tau^* \geq \left(\int_V \tau^k \,\mathrm{d}V \right)^{\frac{1}{k}} = \left(3.3 \cdot \frac{A}{A^k} \cdot \int_0^L V(x)^k \,\mathrm{d}x \right)^{\frac{1}{k}} \tag{21}$$

$$\tau^* \geq 1.27 \cdot A^{\frac{1-k}{k}} \cdot \left(\int_0^L V(x)^k \,\mathrm{d}x \right)^{\frac{1}{k}} \tag{22}$$

Determination of Shape and Scale Parameters

The shape parameter, k, and the scale parameter, m, are determined from tested beams with known volume, shear force diagram, and shear failure load. To calculate k and m the stress at a reference location in the beam due to the failure load is first computed. The maximum shear stress in the cross-section is typically selected, which for rectangular cross-sections read

$$\tau_M = \frac{3V}{2A} \tag{23}$$

Next, this reference stress enters a generic equation that expresses the stress at any location in the beam:

$$\tau(x,y,z) = \tau_M \cdot \theta(x,y,z) \tag{24}$$

Substitution into Eq. (9) yields

$$\int_V (\tau_M \cdot \theta)^k \, \mathrm{d}V = \left(\tau^*\right)^k \quad \Rightarrow \quad \tau_M = \frac{\tau^*}{\left(\int_V \theta^k \, \mathrm{d}V\right)^{\frac{1}{k}}} \tag{25}$$

Factorize out the shear force from the integral and introduce I(k), and equivalently β:

$$\tau_M = \frac{\tau^*}{\left(I(k) \cdot V\right)^{\frac{1}{k}}} = \beta(k) \cdot \frac{\tau^*}{\left(V\right)^{\frac{1}{k}}} \tag{26}$$

In a plot with $\ln(\tau_M)$ along the abscissa axis versus $\ln(V)$ along the ordinate axis the slope of the function is $1/k$. To ease the regression, the median τ_M for each beam configuration is usually plotted. Knowing k, β, and the median values of τ_M, the corresponding media values of τ^* are computed from Eq. (25). This gives the median value of the Weibull-distributed random variable τ^*, which corresponds to the inverse CDF value at 0.5 probability of exceedance

$$0.5 = 1 - e^{-\left(\frac{\tau^*_{\text{median}}}{m}\right)^k} \tag{27}$$

Solving for m yields:

$$m = \frac{\tau^*_{\text{median}}}{\left(-\ln(1-0.5)\right)^{1/k}} \tag{28}$$

Regression results from Foschi and Barrett for 1.0m^3 reference volume is k=5.53 and m=2,540kN/m^2. The values for k and m are substituted into the Weibull equation, i.e., Eq. (8), to obtain reference strength values for different exceedance probabilities:

$$\tau^*(p) = m\left(-\ln(1-p)\right)^{1/k} \tag{29}$$

For example:

$$\begin{aligned}\tau^*_{0.5} &= (2{,}540\text{kN/m}^2)\cdot\left(-\ln(1-0.5)\right)^{1/5.53} = 2{,}377\text{kN/m}^2 \\ \tau^*_{0.05} &= (2{,}540\text{kN/m}^2)\cdot\left(-\ln(1-0.05)\right)^{1/5.53} = 1{,}484\text{kN/m}^2\end{aligned} \tag{30}$$

References

It would be appropriate to include references here to Weibull as well as work by Dr. Richardo Foschi at UBC and others.