Fatigue

Stress Concentrations

A fundamental topic in fatigue is the development of cracks in steel. For structures subjected to cyclic loads, such cracks can develop over time and ultimately reach the size at which some dramatic failure occur. While this problem is rare in structures made of "mild steel," the use of "high tensile" steel changes that situation. It is straightforward to double the strength of mild steel by adding carbon, but this may increase the material's brittleness by a factor of fifteen (Gordon 1978). If the stress in the structure was also doubled to take advantage of the increased strength, then the capacity of the material to cope with cracks is reduced by a factor of as much as $2 \cdot 2 \cdot 15=60$, as explained by the equations in this document. This is a dramatic reduction which may make cracks the governing design concern.

The unsuspecting steel engineer may think that the stresses calculated by the theory of elasticity for beams and plates can simply be compared with the yield stress to tell if the structure will stand or fall. However, two additional concerns must often be addressed. One is related to compressive stresses, where the stability against buckling must be considered. The other is related to tensile stresses and is addressed in this document. The ultimate cause of failure is severe cracks, but the underlying cause is stress concentrations. Specifically, the stresses computed by the theory of elasticity increase multi-fold at sharp corners, initial cracks in the material, and other discontinuities. These high and localized stresses can cause cracks to form and grow in an uncontrolled fashion.

The stress increase in the vicinity of some discontinuity is usually expressed by a "stress concentration factor." This factor is multiplied by the average tensile stress in the surrounding area to obtain the peak stress. The stress concentration factor for a crack of length L and tip-radius r is (Inglis 1913):

$$K = 1 + 2 \cdot \sqrt{\frac{L}{r}} \tag{1}$$

It is observed that when $L=r$, i.e., when the crack is semi-circular, the stress concentration factor is 3. Sharper cracks leads to higher stress concentration.

Fracture Mechanics

Fracture mechanics is an analytical approach to the study of crack propagation. In contrast, the subsequent section on fatigue gives an empirical alternative to determine the number of stress cycles that can be handled before a crack grows out of control. The concept of energy is important in fracture mechanics. In particular, the energy needed to grow a crack is compared with the energy provided by the surrounding material. The former is proportional to $W \cdot L$, where W is the "work of fracture" of the material and L is the crack length. Conversely, the energy released around the crack is proportional to L^2, which is understood by considering triangular areas on either side of the crack. These two terms are illustrated in Figure 1 (Gordon 1978). Because the energy needed to grow the crack varies linearly with L and the energy given by the surrounding material varies quadratically with L there will be a crack length at which the crack starts growing uncontrollably. This crack length is denoted L_g in the figure. The subscript on L_g is from Griffith, who developed the following formula for the "critical crack length" (Griffith 1921):

$$L_g = \frac{2 \cdot W \cdot E}{\pi \cdot s^2} \tag{2}$$

where W=work of fracture, E=Young's modulus, and s=average tensile stress in the surrounding area without stress concentration included.

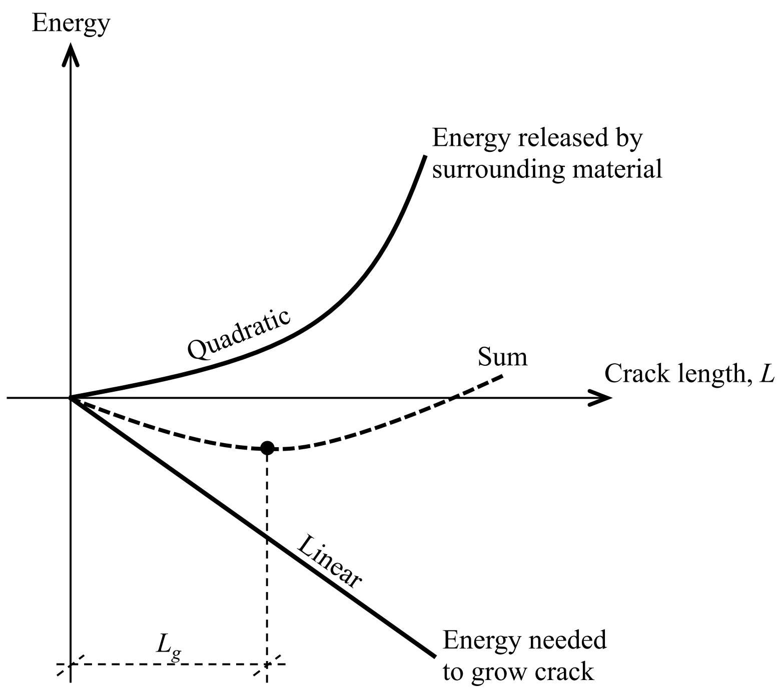

Figure 1: Energy balance in crack growth.

The problem of fatigue, namely the potential growth of cracks due to a high number of load cycles, is an important design concern, particularly for steel structures. Fatigue may occur in many structural applications, from bridges to ships. For example, the number of wave-induced bending cycles of a typical ship may be in the order of 10^8 during a 20-year time period (Hughes and Paik 2010). In addition comes cycles due to loading and unloading, as well as engine and propeller vibration.

Fracture mechanics represents an analytical approach to address fatigue, i.e., crack growth due to cyclic loading. The other approach is to use empirical S-N curves, which is addressed in the next section. In fracture mechanics, the crack growth is formulated by means of a damage accumulation model, such as:

$$\frac{da}{dN} = C \cdot (\Delta K)^m \tag{3}$$

where a is the crack length, N is number of load cycles, C and m are material constants, and ΔK is the range of the stress intensity factor K, which represents the stress range. Techniques to solve for a are presented in another document on damage accumulation models.

S-N Curves and Miner's Rule

The alternative to fracture mechanics for the consideration of fatigue is the use of S-N curves and an assumption called Miner's rule. The S-N diagrams simply display the number of load cycles, N, at a constant stress amplitude, S, it took for a test-specimen to fail in fatigue. A schematic S-N diagram is shown in Figure 2. It is observed that the horizontal distance to the S-N curve is the "fatigue life" of the test-specimen at a given stress level, and that below a certain stress level there is zero damage accumulation. That stress level is denoted S_∞ and is called the "fatigue limit." The sloped line in Figure 2 is straight because S-N diagrams are presented in a log-log plot. In fact, the SN curve is expressed as

$$S(N) = \left(\frac{K}{N} \right)^{\frac{1}{m}} \tag{4}$$

which implies that

$$\log(N) = \log(K) - m \cdot \log(S) \tag{5}$$

which reveals that m is the negative slope of the S-N curve in the log-log plot while K is another constant that essentially represents the intercept. It is noted from Eq. (5) that the S-N curve can also be written

$$N = \frac{K}{S^m} \tag{6}$$

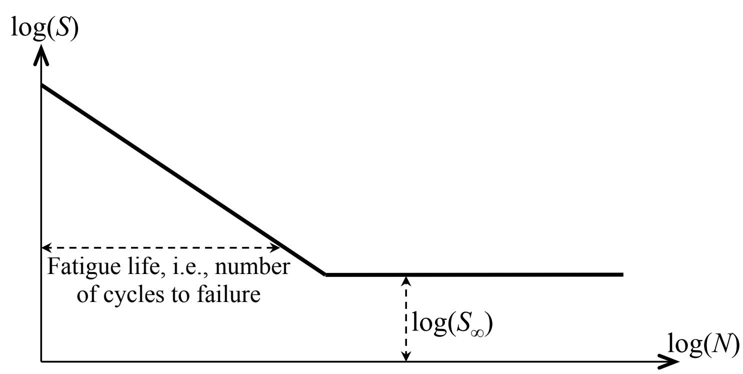

Figure 2: Schematic S-N curve.

The S-N curves described above appear to account only for the stress range and not the average stress. This is not a problem when the stress-cycles oscillates around zero, but could be an issue when the average stress is high, with cycles on top of that. One approach to include the mean stress is the "Goodman correction," which assumes that the damage done by stress cycles with mean S_{mean} and range S_{range} is the same as an auxiliary stress process with zero mean and range S_{aux}, related by the equation (Lutes and Sarkani 1997)

$$\frac{S_{range}}{S_{aux}} + \frac{S_{mean}}{f_u} = 1 \tag{7}$$

where f_u is the ultimate stress capacity of the material. S_{aux} is solved from Eq. (7) and used with the ordinary S-N diagram above. An alternative to the Goodman formulation above is the Gerber formula:

$$\frac{S_{range}}{S_{aux}} + \left(\frac{S_{mean}}{f_u}\right)^2 = 1 \tag{8}$$

Palmgren-Miner's Rule

In practice, S-N diagrams are applied in conjunction with Miner's rule, which states that damage accumulates linearly, in the sense that the fatigue life at different stress levels can be added. As a result, the stress amplitudes that the structure experiences are sorted in bins, where ΔS_i and n_i are the stress range and number of cycles in that stress range, respectively. The total fatigue damage is then measured as

$$D = \sum_{i=1}^{Bins} \frac{n_i}{N_i} \tag{9}$$

Theoretically, $D<1$ means the structure is safe, while $D>1$ implies fatigue failure. In practice, D is not allowed anywhere near unity. Rather, D is typically limited to 0.1 to 0.3, depending on how easy it is to inspect and detect potential damage during the service life of the structure.

Rainflow Count of Cycles

For complicated stress histories it is non-trivial to count stress cycles in order to apply S-N diagrams. Hence, the problem addressed in this section is how to divide a complicated time-history into cycles, to count those cycles, and then to determine the damage, D. The adopted approach is referred to as the rainflow method, which can be interpreted in several ways. For example, by Masing's hypothesis it can be linked with hysteretic stress-strain curves, in which one rainflow cycle is one closed stress-strain hysteresis loop. Other interpretations are also possible.

It is an objective to count all cycles, small and large, including the large ones that are interrupted by small cycles. The rainflow method counts half-cycles, and every part of the time-history are associated with exactly one of the identified half-cycles. At each local peak or local valley, one half-cycle is ending and another is starting. Importantly, the peaks and valleys are paired so as to give the largest possible half cycle, going from the biggest possible to the next.

The rainflow method of counting cycles is visualized in Figure 3 for two sample time histories. It is first observed that the time histories are turned 90° to facilitate the rainflow analogy. As a result, the dashed lines in Figure 3 can be thought of as rainflows on a pagoda roof. The time-history in Figure 3a is periodic with constant amplitude. A quick count without any rainflow analogy reveals that it contains four full cycles. In contrast, there are eight rainflows that originate at each extreme. Each of these rainflows represents a half-cycle. As a result, half-cycle count by the rainflow method for the time-history in Figure 3a yields eight half-cycles. Four of the half-cycles are in tension, terminating with an arrow on the right side, and four are in compression, terminating with an arrow on the left side. In this case, all the half-cycles are of equal magnitude, hence the four tension-half-cycles can be added to the four compression-half-cycles to yield four full stress cycles, which is obviously the correct answer.

The time-history in Figure 3b is more complicated. According to the rainflow method, each compression half-cycle starts at a tension peak and ends when it meets a flow from above or when it falls all the way to the ground, so to speak. With that in mind, there are four compression half-cycles, named 1, 2, 3, and 4 in Figure 3b, and there are four tension half-cycles, named A, B, C, and D. The magnitude of each half cycle is the stress range that it has travelled, i.e., horizontal distance. In this case the half cycles are all of different magnitude, thus they cannot directly be added to obtain full cycles. It is often the case in rainflow analysis that some "spare" half-cycles remain after the half-cycles have been added to a number of full cycles.

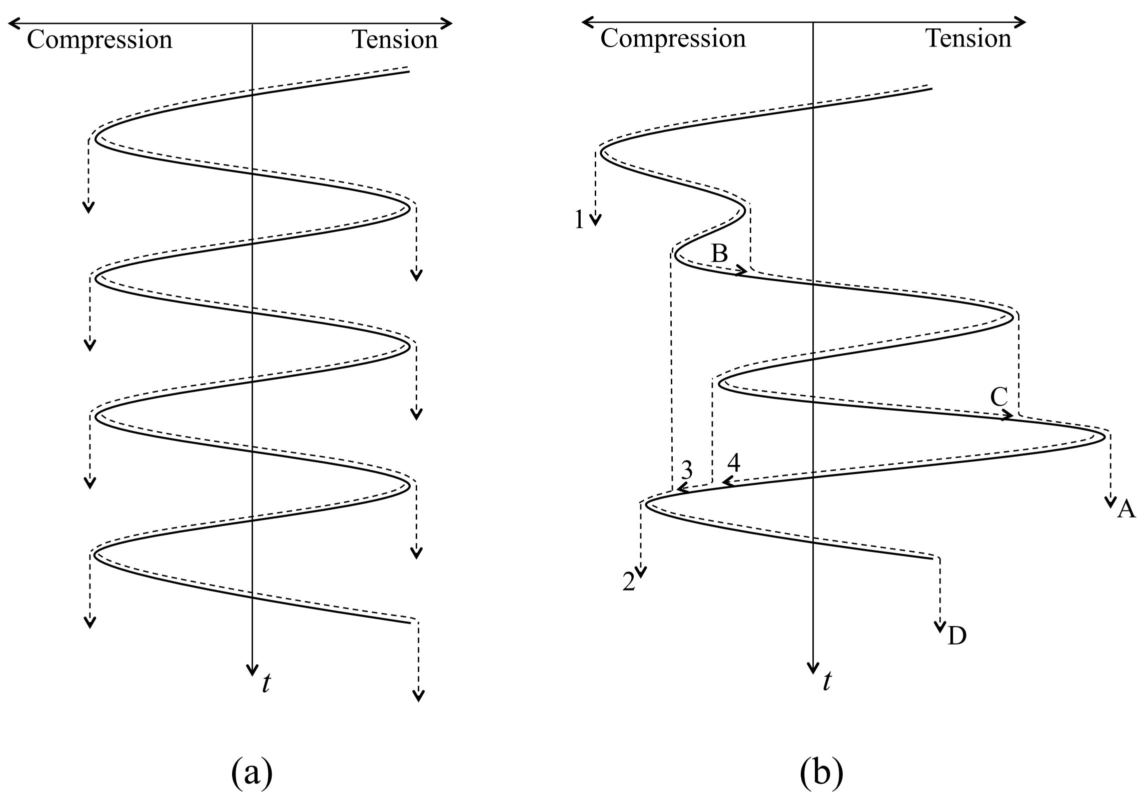

Figure 3: Counting half-cycles by the rainflow method.

Several algorithms exist for counting half-cycles by the rainflow method. An algorithm proposed by Downing and Socie in 1982 is particularly popular, and is provided in the following (Downing and Socie 1982; Lutes and Sarkani 1997). The algorithm starts by reorganizing the time-history slightly; the start-time is taken as the location of the highest peak, and the time-history that preceded that peak is moved to the end. Next, the amplitude of all the local peaks and valleys are collected in the vector **x**, obviously with the first element being largest because of the mentioned reorganization of the time-history. During the execution of the algorithm, a vector **q**, which varies in size, is also maintained. The algorithm is initialized by setting $q_1=x_1$, $q_2=x_2$, $q_3=x_3$, and $n=3$, and then repeats the following steps until there is no more data in **x**, i.e., until x_m is out of bounds:

1. Set range: $R_1=|q_n-q_{n-1}|$
2. Set previous adjacent range: $R_2=|q_{n-1}-q_{n-2}|$
3. If ($R_1<R_2$) then R_2 is NOT a rainflow range, so do the following:
    a. $n=n+1$
4. Else R_2 is a rainflow range and do the following:
    a. Register R_2 as a rainflow half-cycle
    b. Remove its two extrema q_{n-1} and q_{n-2} from **q**, i.e., shorten **q** by two
    c. $n=n-2$
    d. Append to **q** the next element of **x** if the size of **q** is less than three

Stochastic Fatigue

The objective now is to estimate the fatigue life of a structure, when the loading is a continuous stochastic process. To address this problem, the concepts of S-N curves and Miner's rule from the document on deterministic fatigue are applied in the context of a stochastic stress history. This helps simplify the problem, but it is still a challenging one, lacking general closed-form solutions.

Rayleigh Approximation

According to Miner's rule, fatigue damage is measured by D and defined by

$$D=\sum_{i=1}^{B}\frac{n_i}{N_i} \tag{10}$$

which steadily increases from zero to unity, at which failure occurs. In Eq. (10), B is the number of bins or stress ranges, n_i is the number of cycles in each stress range, and N_i is the number of cycles that causes failure in in that range. In stochastic fatigue it is helpful to think of a damage-increment, ΔD, which adds to the damage in each stress cycle. Of course, each cycle may have a different damage-increment, but for a given stress range it is assumed that ΔD is the same in each cycle. In other words, the damage-increment in a cycle is uniquely defined by the stress range of that cycle. According to Eq. (10), the damage-increment in each cycle is

$$\Delta D_i=\frac{1}{N_i} \tag{11}$$

where ΔD_i and N_i are the damage-increment and number of cycles to failure at stress range number i. With that definition of ΔD_i it is possible to deal with situations with different stress range in each cycle. In that case there are as many bins as the number of cycles, and failure occurs when

$$D = \sum_{i=1}^{N_f} \Delta D_i = 1 \tag{12}$$

where N_f is the total number of stress cycles to failure, each cycle with its own stress range. Eq. (12) contains a sought quantity, namely N_f, a measure of the fatigue life of the structure. To estimate this quantity, the expectation of Eq. (12) is considered:

$$E[D] = \sum_{i=1}^{N_f} E[\Delta D] = E[N_f] \cdot E[\Delta D] = 1 \tag{13}$$

Solving for E[N_f] yields

$$E[N_f] = \frac{1}{E[\Delta D]} \tag{14}$$

From the total expected number of cycles, E[N_f], the expected fatigue life is obtained by multiplying by the duration of each cycle. The average duration of the cycles is the inverse of the average rate of occurrence of cycles. For a Gaussian process that rate is approximated by the up-crossings of the mean stress, which is

$$\nu_U^+(\mu_U) = \frac{1}{2\pi} \cdot \sqrt{\frac{\lambda_2}{\lambda_0}} \tag{15}$$

As a result, the fatigue life estimate is the expected number of cycles multiplied by the average cycle duration:

$$E[T] = E[N_f] \cdot \frac{1}{\nu_U^+(\mu_U)} = \frac{2\pi}{E[\Delta D]} \cdot \sqrt{\frac{\lambda_0}{\lambda_2}} \tag{16}$$

The remaining task is to address the damage-increment. To link the amplitude of a stress cycle to a particular damage increment, the generic expression for an S-N curve (those are obtained from experimental data and introduce the "capacity" of the material) is substituted into Eq. (11):

$$\Delta D_i = \frac{1}{N_i} = \frac{1}{\left(\frac{K}{S_i^m}\right)} = \frac{S_i^m}{K} \tag{17}$$

The expectation, needed in Eq. (16), is

$$E[\Delta D] = \frac{E[S^m]}{K} \tag{18}$$

In what is referred to as the Rayleigh approximation of stochastic fatigue analysis, it is here assumed that the stress peaks are Rayleigh distributed, which is appropriate for Gaussian processes. As a result, the m^{th} moment of the stress range, estimated as two times the peak, u_p, is

$$
\begin{aligned}
E[S^m] = E[(2u_p)^m] &= \int_0^\infty (2u_p)^m \cdot f_{u_p}(u_p)\,du_p \\
&= 2^m \cdot \int_0^\infty \frac{u_p^{m+1}}{\sigma_U^2} \cdot \exp\left(-\frac{u_p^2}{2 \cdot \sigma_U^2} \right) du_p \\
&= 2^{1.5m} \cdot \sigma_U^m \cdot \Gamma\left(1 + \frac{m}{2} \right) \\
&= 2^{1.5m} \cdot \lambda_0^{0.5m} \cdot \Gamma\left(1 + \frac{m}{2} \right)
\end{aligned} \tag{19}
$$

In summary, the expected fatigue life is

$$
E[T] = \frac{2\pi \cdot K \cdot 2^{-1.5m} \cdot \lambda_0^{0.5\cdot(1-m)} \cdot \lambda_2^{-0.5}}{\Gamma\left(1 + \frac{m}{2} \right)} \tag{20}
$$

Stochastic Rainflow Analysis

The rainflow method for counting half-cycles is described in the document on deterministic fatigue considerations. One brute-force application of this method to stochastic fatigue is to simulate realizations of the stochastic stress process, followed by rainflow counting. Like Monte Carlo sampling, that approach is straightforward but accurate results come at a high computational cost. A special case that facilitates an analytical solution is when the S-N curve is described by m=1 (Lutes and Sarkani 1997). In that case, the damage increments are simply

$$
\Delta D_i = \frac{S_i}{K} \tag{21}
$$

As a result, the failure criterion is

$$
D = \sum_{i=1}^{N_f} \Delta D_i = \frac{1}{K} \cdot \sum_{i=1}^{N_f} S_i = 1 \tag{22}
$$

In this case, the rainflow sum of stress ranges is obtained by adding contributions from each infinitesimal time increment dt. Denoting the stress process by $U(t)$, the increment of stress during dt is $\dot{U}(t) \cdot dt$. Each of these stress increments is part of a stress range counted by the rainflow method. Because the stress increment can be positive or negative, the absolute value is introduced to obtain the sum:

$$
\sum_{i=1}^{N_f} S_i = \frac{1}{2} \cdot \int_0^T \left| \dot{U}(t) \right| dt \tag{23}
$$

where the factor ½ is introduced because an actual full stress cycle consists of *two* increment paths, i.e., one towards more tension and the other towards more compression. The expected value of the integral is

$$E\left[\int_0^T \left|\dot{U}(t)\right| dt\right] = E[T] \cdot E\left[\left|\dot{U}(t)\right|\right] \tag{24}$$

Substitution of the expectation on the left-hand side of Eq. (24) for the integral in Eq. (23) and substitution of the sum in Eq. (23) into Eq. (22) yields:

$$E[T] = \frac{2K}{E\left[\left|\dot{U}(t)\right|\right]} \tag{25}$$

For a Gaussian process, the derivative process has the normal distribution

$$f_{\dot{U}}(u) = \frac{1}{\sqrt{2\pi} \cdot \sigma_{\dot{U}}} \cdot \exp\left(-\frac{1}{2} \cdot \left(\frac{u}{\sigma_{\dot{U}}}\right)^2\right) \tag{26}$$

which means that the expectation of the absolute value in Eq. (25) is

$$E\left[\left|\dot{U}(t)\right|\right] = \int_0^\infty u \cdot f_{\dot{U}}(u) du = \sqrt{\frac{2}{\pi}} \cdot \sigma_{\dot{U}} \tag{27}$$

Substitution of Eq. (27) into Eq. (25) yields

$$E[T] = \frac{\sqrt{2\pi} \cdot K}{\sigma_{\dot{U}}} \tag{28}$$

This result can be compared with the Rayleigh approximation presented earlier by setting m=1 and $\lambda_2 = \sigma_{\dot{U}}^2$ in Eq. (20), which yields

$$E[T] = \frac{2\pi \cdot 2^{-1.5} \cdot K}{\Gamma(1.5) \cdot \sigma_{\dot{U}}} = \frac{\sqrt{2\pi} \cdot K}{\sigma_{\dot{U}}} \tag{29}$$

This coincidence of the rainflow approach with the Rayleigh approximation is due to the selection of m=1 in the S-N curve, but did not include any assumption on the bandwidth.

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