Damage Accumulation Models

A Creep Model

Creep is a long-term effect in structural materials, by which deformations slowly increase until possible failure. The phrase "duration-of-load" is often applied to the phenomenon because it captures the two key design variables of the problem: How much load is applied for how long. In fact, a simple visualization of the capacity of a material to sustain load over long time is shown in Figure 1. Failure due to creep is called "creep rupture" and Figure 1 is referred to as a "creep rupture curve."

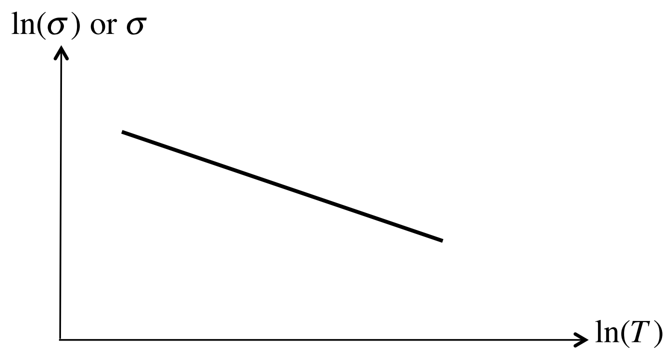

Figure 1: Relationship between stress and time to creep rupture.

Creep rupture curves like the one in Figure 1 are often entirely empirical, but there also exist damage accumulation models that model the phenomenon. One model is (Barrett and Foschi 1978)

$$\frac{d\alpha}{dt} = \begin{cases} a \cdot (\sigma - \sigma_0)^b \cdot \alpha^c & \text{if } \sigma > \sigma_0 \\ 0 & \text{if } \sigma \leq \sigma_0 \end{cases} \tag{1}$$

where α is the measure of accumulated damage (α=0 denotes no damage while α=1 denotes rupture) and σ_0 is the stress below which damage is assumed not to accumulate. The parameters *a*, *b,* and *c* are calibration constants. In the following, consider the case $\sigma>\sigma_0$ and rearrange Eq. (1) to the form:

$$\alpha^{-c} \cdot \frac{d\alpha}{dt} = a \cdot (\sigma - \sigma_0)^b \tag{2}$$

Next, this equation is integrated from t=0 to t=T. To accomplish this it is necessary to apply integration by substitution to the left-hand side, which because $\alpha(0)$=0 evaluates to:

$$\int_0^T \alpha^{-c} \cdot \frac{d\alpha}{dt} dt = \int_{\alpha(0)}^{\alpha(T)} x^{-c}\, dx = \frac{1}{1-c} \cdot \alpha(T)^{(1-c)} \tag{3}$$

This left-hand side together with the integral of the right-hand side yields the following solution to Eq. (2), i.e., the damage, $\alpha(T)$, at time T:

$$\frac{1}{1-c} \cdot \alpha(T)^{(1-c)} = \int_0^T a \cdot \left(\sigma - \sigma_0\right)^b dt \tag{4}$$

Rearranging and acknowledging the dependence of the stress on time yields the following solution for the evolution of creep:

$$\alpha(T)^{(1-c)} = a \cdot (1-c) \cdot \int_0^T \left(\sigma(t) - \sigma_0\right)^b dt \tag{5}$$

Creep rupture occurs when α=1. According to Eq. (5) this leads to the following equation from which the time, T, until rupture can be solved:

$$1 = a \cdot (1-c) \cdot \int_0^T \left(\sigma(t) - \sigma_0\right)^b dt \tag{6}$$

For example, if the stress is constant with time then the time until creep rupture is:

$$T = \frac{1}{a \cdot (1-c) \cdot \left(\sigma - \sigma_0\right)^b} \tag{7}$$

Another damage accumulation model for creep is (Barrett and Foschi 1978):

$$\frac{d\alpha}{dt} = \begin{cases} a \cdot \left(\sigma - \sigma_0\right)^b + \lambda \cdot \alpha & \text{if } \sigma > \sigma_0 \\ 0 & \text{if } \sigma \leq \sigma_0 \end{cases} \tag{8}$$

Again the case $\sigma > \sigma_0$ is considered and the model is an ordinary linear inhomogeneous differential equation of the first order:

$$\frac{d\alpha}{dt} - \lambda \cdot \alpha = a \cdot \left(\sigma - \sigma_0\right)^b \tag{9}$$

To solve this differential equation its characteristic equation is established:

$$\gamma - \lambda = 0 \tag{10}$$

The solution is λ and the homogeneous solution is:

$$\alpha(t) = C_1 \cdot e^{\lambda \cdot t} \tag{11}$$

Together with the particular solution, the damage evolution is described by:

$$\alpha(t) = e^{\lambda \cdot t} \cdot \int_0^T a \cdot \left(\sigma(t) - \sigma_0\right)^b \cdot e^{-\lambda \cdot t} dt \tag{12}$$

A Crack Growth Model

One model for fatigue crack propagation is (Joint Committee on Structural Safety 2001)

$$\frac{da}{dn} = C \cdot Y(a) \cdot \left(\Delta S(n) \cdot \sqrt{\pi \cdot a}\right)^m \tag{13}$$

where $a(n)$ is the crack size, n is number of cycles, C is a material constant, $Y(a)$ is a geometry-dependent function, $\Delta S(n)$ is the stress range, and m is another material constant. Rearranging yields

$$a^{-\frac{m}{2}} \cdot \frac{da}{dn} = C \cdot Y(a) \cdot \left(\Delta S(n) \cdot \sqrt{\pi} \right)^m \tag{14}$$

To integrate this equation from 0 to N, integration by substitution is first applied to the left-hand side:

$$\int_0^N a^{-\frac{m}{2}} \cdot \frac{da}{dn} dn = \int_{a(0)}^{a(N)} x^{-\frac{m}{2}} dx = \frac{2}{2-m} \cdot a(N)^{(2-m)/2} \tag{15}$$

where it was assumed that $a(0)$=0. Integration of the right-hand side of Eq. (14), assuming constant stress range and constant geometry factor, yields:

$$\int_0^N C \cdot Y(a) \cdot \left(\Delta S(n) \cdot \sqrt{\pi} \right)^m dn = C \cdot Y \cdot \Delta S^m \cdot \pi^{m/2} \cdot N \tag{16}$$

Combining the left-hand side and the right-hand obtained above yields the following integrated solution to Eq. (14), i.e., the crack size, $a(N)$, at cycle N:

$$\frac{2}{2-m} \cdot a(N)^{(2-m)/2} = C \cdot Y \cdot \Delta S^m \cdot \pi^{m/2} \cdot N \tag{17}$$

Suppose failure occurs if the crack size exceeds a critical threshold, a_{cr}, Eq. (17) can be rearranged to express failure as follows:

$$\frac{1}{C \cdot Y} \cdot \pi^{-m/2} \cdot \frac{2}{2-m} \cdot a(N)^{(2-m)/2} \leq \Delta S^m \cdot N \tag{18}$$

Another model for crack growth, formulated as a function of time instead of number of cycles, is

$$\frac{da}{dt} = A \cdot K^m(t) \tag{19}$$

where

$$K(t) = \sigma(t) \cdot \sqrt{a} \tag{20}$$

Following the solution procedure above, rearranging yields:

$$a^{-m/2} \cdot \frac{da}{dt} = A \cdot \sigma^m \tag{21}$$

Integration with respect to time from 0 to T, with application of integration by substitution to the left-hand side, yields:

$$\frac{2}{2-m} \cdot a(T)^{(2-m)/2} = A \cdot \int_0^T \sigma(t)^m dt \tag{22}$$

from which the crack size, a, can be solved for.

References

Barrett, J. D., and Foschi, R. O. (1978). "Duration of load and probability of failure in wood. Part I. Modelling creep rupture." *Canadian Journal of Civil Engineering*, 5(4), 505–514.

Joint Committee on Structural Safety. (2001). *Probabilistic Model Code*.