Continuous Stochastic Processes

The term "stochastic" is often applied to phenomena that vary in *time*, while the word "random" is reserved for phenomena that vary in *space*. Apart from this distinction, the modelling of uncertain temporal and spatial variation is quite similar. In fact, the concepts described in the following for stochastic processes remain valid also for uniaxial random fields. The differences that exist relate to terminology. For example, a *stationary* stochastic process is referred to as *homogeneous* in the context of random fields.

A stochastic process is essentially a collection of random variables along the time axis. As an illustration, consider the temporal variation of, say, the wind pressure intensity at a particular location of a building. The intensity at a specific time instant is an individual random variable. The intensity at a later time instant is another random variable, often correlated with the first if they are close in time. It is understood that because there are infinitely many time-instances within even a short interval there is also an infinite number of random variables in a process. That does not cause any conceptual problems; there is no need to enumerate the random variables until realizations are generated, and then the problem is a practical matter of granularity rather than a conceptual problem.

If the random variables of a process each have a continuous probability distribution, then the process is said to be continuous, otherwise it is discrete and outside the scope of this document. The present document explains continuous processes, which are employed to model continuously varying loads, such as ocean waves and wind, and even the ensuing response of the structure.

Time Domain Model Description

Consider a continuous stochastic process, $X(t)$. Similar to the notation for random variables, the realizations are denoted by the corresponding lowercase letter, i.e., $x(t)$. A group of realizations if often referred to as an ensemble. It is understood from above that $X(t)$ is a collection of random variables $X(t_i)$, where t_i are infinitely many time instants. This family of random variables is sometimes written $\{X(t)\}$ (Lutes and Sarkani 1997). On that basis it is clear that the model for a stochastic process is essentially a joint probability distribution for those random variables. This joint distribution can be written as a CDF, CCDF, or as the joint PDF

$$f(x,t) = f_{X(t_1),X(t_2),\cdots,X(t_n)}\left(x(t_1),x(t_2),\cdots,x(t_n)\right) \tag{1}$$

An abbreviated version of Eq. (1) is obtained by replacing $X(t_i)$ with X_i, which yields

$$f(x,t) = f_{X_1,X_2,\cdots,X_n}\left(x_1,x_2,\cdots,x_n\right) \tag{2}$$

Mean Function and Second-Moment Functions

The joint distribution tells the full statistical story of a stochastic process. However, in the same way as statistical moments are helpful to describe random variables, partial descriptors are employed also to describe a stochastic process. In this context, the first statistical moment corresponds to the mean function:

$$\mu_X(t) = E\left[X(t)\right] = \int_{-\infty}^{\infty} x(t)\cdot f(x,t)\,dx \tag{3}$$

Now moving to second moments, it is the dependence between the process at two different time-instants that is of interest. To prepare for these considerations, it is first noted that there are several ways to express the covariance between two random variables X_1 and X_2: covariance, Cov[X_1,X_2], correlation coefficient, ρ_{12}, and mean product, E[X_1X_2]. From the document on multivariate distributions, the relationship between these quantities is

$$Cov\left[X_1,X_2\right] = E\left[X_1X_2\right] - \mu_1\mu_2 = \rho_{12}\cdot\sigma_1\cdot\sigma_2 \tag{4}$$

where μ_i are the means and σ_i are the standard deviations of X_1 and X_2. Turning to stochastic process, an important second-moment descriptor is the autocorrelation function

$$\phi_{XX}(t_1,t_2) = E\left[X(t_1)\cdot X(t_2)\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1\cdot x_2\cdot f(x,t)\,dx_1\,dx_2 \tag{5}$$

For second-moment stationary processes, described shortly, it is possible to reduce the two time-instants t_1 and t_2 in the argument to one parameter $\tau=|t_1-t_2|$, namely the temporal distance between two points anywhere along the time axis. In that case the notation is changed from ϕ to R and the autocorrelation function reads

$$R_{XX}(\tau) = E\left[X(t)\cdot X(t+\tau)\right] \tag{6}$$

where τ=0 gives the mean square function E[$X(t)^2$]. While the interpretation of the mean function in Eq. (3) is conceptually straightforward, the autocorrelation function warrants comments about its name, as well as its meaning. The name "auto" implies correlation within one process, to distinguish it from the cross-correlation, R_{XY}, between two different processes $X(t)$ and $Y(t)$. The meaning of autocorrelation is approached in several ways. First, it is understood from Eqs. (5) and (6) that autocorrelation corresponds to the concept of "mean product" for random variables. In other words, the autocorrelation function does not display correlation, but Eq. (4) does reveal that it is related to the concept of correlation.

An alternative second-moment descriptor of a stochastic process is the autocovariance function, which corresponds directly to the covariance concept for random variables:

$$\kappa_{XX}(t_1,t_2) = E\left[\left(X(t_1)-\mu_x(t_1)\right)\cdot\left(X(t_2)-\mu_x(t_2)\right)\right] \tag{7}$$

Again second-moment stationarity allows an alternative notation in terms of one parameter:

$$C_{XX}(\tau) = E\left[\left(X(t)-\mu_x(t)\right)\cdot\left(X(t+\tau)-\mu_x(t+\tau)\right)\right] \tag{8}$$

where τ=0 gives the variance function:

$$C_{XX}(0) = E\left[\left(X(t)-\mu_x(t)\right)^2\right] = \sigma_X^2(t) \tag{9}$$

It is common to work with zero-mean processes. In fact, if a process X(t) has some constant mean μ_X then it is easily transformed into the zero-mean process X(t)–μ_X. A glance at Eq. (4) reveals that $C_{XX}(\tau)=R_{XX}(\tau)$ for zero-mean processes, thus the autocorrelation function and autocovariance function can be used interchangeably for such processes.

As a matter of completeness, it is a small step to employ Eq. (4) to express the autocorrelation coefficient function of a stochastic process:

$$\rho_{XX} = \frac{C_{XX}(\tau)}{\sqrt{C_{XX}(0)}\cdot\sqrt{C_{XX}(0)}} = \frac{C_{XX}(\tau)}{C_{XX}(0)} \qquad (10)$$

Above, the three second-moment descriptor functions were introduced for dependence within a stochastic process. To obtain the corresponding expressions for cross-correlation, cross-covariance, and cross-correlation coefficient between two processes $X(t)$ and $Y(t)$ it is sufficient to replace the indices XX in all expressions with XY.

Stationarity

Roughly speaking, a process that is stationary has properties that do not change along the time axis. Several types of stationarity are defined:

- "Mean-value stationarity" ....the mean is constant
- "Second-moment stationarity" ....the autocorrelation function is constant
- "Covariant stationarity" ....the autocovariance function is constant
- "n^{th}-moment stationarity" ....the n^{th} moment is constant
- "n^{th}-order stationarity" ....the joint probability distribution for the process evaluated at n points is invariant to a time-shift
- "Strict stationarity" ....all properties of the process are constant
- "Weak stationarity" ....is not uniquely defined, but often implies mean-value and second-moment stationarity

Continuity

The concept of continuity appears in several contexts for stochastic processes, and some have been alluded to already. It is already understood that the processes considered in this document are continuous along the time axis, and that the probability distribution for the random variable $X(t)$ at any time instant is continuous. Another consideration is the continuity of the autocorrelation and autocovariance functions. It can be shown that second-moment stationary processes with continuous autocorrelation and autocovariance functions at τ=0 also must be continuous for all other values of τ. Additional considerations of continuity for autocorrelation and autocovariance functions for non-stationary processes are made in the plane stretched by t_1 and t_2, yet to be discussed in this document. Several other continuity requirements can be also formulated, but it is for now assumed in this document that all the realizations and associated functions are continuous.

Ergodicity

An ergodic process has the advantageous property that one can average over time to infer the mean, autocorrelation function, and other quantities, instead of averaging over an ensemble of realizations. Underneath the concept ergodicity is a form of statistical independence in time. Naturally, ergodicity implies stationarity, and there are as may types of ergodicity as there is stationarity. However, stationarity does NOT imply ergodicity.

Frequency Domain Model Description

The autocorrelation function, described above, says something about how rapidly the amplitude of a process varies in time. Specifically, if $R_{XX}(\tau)$ diminishes rapidly with τ then there is little correlation between the amplitudes even short times apart, and realizations will look disorderly with rapid variations in amplitude. In contrast, if $R_{XX}(\tau)$ diminishes slowly with τ then correlation is high between the amplitude at two time-instant even when the times are far apart, which means that the amplitude is slowly varying in time. Another way to look at this is to consider frequencies, which is the topic in this section. It is intuitive that a disorderly realization has contributions from many frequencies, while a slowly varying realization has contributions from fewer frequencies. The quantity that captures the frequency content of a process is the power spectral density (PSD), $S_X(\omega)$, which is the Fourier transform of the autocovariance function:

$$S_X(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty}C_{XX}(\tau)\cdot e^{-i\cdot\omega\cdot\tau}\,d\tau=\frac{1}{\pi}\int_{0}^{\infty}C_{XX}(\tau)\cdot e^{-i\cdot\omega\cdot\tau}\,d\tau \tag{11}$$

where the last equality is possible because of the symmetry of $C_{XX}(\tau)$. The autocovariance function and the PSD form a Fourier transform pair; hence, the autocovariance is obtained from the spectral density as

$$C_{XX}(\tau)=\int_{-\infty}^{\infty}S_X(\omega)\cdot e^{i\cdot\omega\cdot\tau}\,d\omega=2\cdot\int_{0}^{\infty}S_X(\omega)\cdot e^{i\cdot\omega\cdot\tau}\,d\omega \tag{12}$$

For completeness it is noted that the non-imaginary form of the PSD in terms of the autocovariance function is

$$\begin{aligned}S_X(\omega)&=\frac{1}{2\pi}\int_{0}^{\infty}C_{XX}(\tau)\cdot e^{-i\cdot\omega\cdot\tau}\,d\tau+\frac{1}{2\pi}\int_{0}^{\infty}C_{XX}(-\tau)\cdot e^{i\cdot\omega\cdot\tau}\,d\tau\\&=\frac{1}{\pi}\int_{0}^{\infty}C_{XX}(\tau)\cdot\left(\cos(\omega\tau)\right)d\tau\end{aligned} \tag{13}$$

which is obtained by first splitting the integral into two symmetric parts, one from 0 to ∞ and the other from 0 to $-\infty$ by the variable change $\tau\to(-\tau)$, followed by substitution of Euler's formula $e^{i\omega\tau}=\cos(\omega\tau)+i\cdot\sin(\omega\tau)$, then using $C_{XX}(\tau)=C_{XX}(-\tau)$ and the facts that $\cos(\omega\tau)=\cos(-\omega\tau)$ and $\sin(\omega\tau)=-\sin(-\omega\tau)$. Similarly, the expression for the autocovariance function in terms of the PSD becomes:

$$C_{XX}(\tau)=2\cdot\int_{0}^{\infty}S_X(\omega)\cdot\cos(\omega\tau)\,d\omega \tag{14}$$

The PSD is the key model descriptor in the frequency domain. An interpretation of the PSD is obtained from Eq. (9), which shows that the variance of a stationary process is:

$$\sigma_X^2 = C_{XX}(0) = \int_{-\infty}^{\infty} S_X(\omega)d\omega = 2 \cdot \int_{0}^{\infty} S_X(\omega)d\omega \tag{15}$$

In words, the area underneath the PSD is the variance of the process. In fact, the PSD essentially displays the amount of variance as function of frequency. That is, the value of the PSD at a particular frequency indicates the relative amplitude of the process at that frequency. Theoretically, the PSD is defined for both positive and negative frequencies, in fact, it is symmetric about ω=0, but negative frequencies is an artificial construct. For that reason it is common to use the "one-sided PSD," which is defined for positive ω only:

$$S_X^+(\omega) = 2 \cdot S_X(\omega) \quad \text{for} \quad \omega \geq 0 \tag{16}$$

which gives the same area underneath the PSD for both $S_X^+(\omega)$ and $S_X(\omega)$. The PSD can be formulated in term of the frequency, f, measured in Hertz, by the transformation ω=2πf. This version of the PSD is denoted $G_X^+(f)$ and is obtained from $S_X^+(\omega)$ by substituting $2\pi f$ for ω, multiplying all ordinate values by 2π, and dividing all abscissa values by 2π. Again, this transformation maintains the area underneath the PSD, i.e., it maintains the variance of the process, which is what the PSD describes.

Measures of Bandwidth

The bandwidth of a process is seen from the width of the PSD. The broader PSD, the more frequencies contribute. A "narrowband process" has a narrow PSD and thus exhibits smoothly varying realizations with nearly just one dominant frequency. Conversely, a "broadband process" has a broad spectrum and realizations that are more chaotic with many contributing frequencies. The extreme-case of broadband processes is the artificial "white noise" process that has a constant spectrum over all frequencies and no autocorrelation.

Several measures exist to characterize the bandwidth of a process. To understand these, it is helpful to compare them with the statistical moments of probability distributions. The moments of the PSD are:

$$\lambda_m = \int_{0}^{\infty} \omega^m \cdot S_X^+(\omega)d\omega \tag{17}$$

The PSD is not directly comparable to a probability distribution because it does not integrate to unity. This property is achieved by rather using the normalized PSD $S_X^+(\omega)/\lambda_0$, where λ_0 is the area underneath the PSD, and the corresponding moments λ_m/λ_0. However, just like the moments of probability distributions, these moments have units and are therefore further normalized to become useful as dimensionless bandwidth measures. Directly analogous to the coefficient of variation of a random variable, a dimensionless bandwidth parameter is

$$
\begin{aligned}
\delta &= \frac{\text{"stdv"}}{\text{"mean"}} = \frac{\sqrt{\text{"mean square"} - \text{"squared mean"}}}{\text{"mean"}} \\
&= \frac{\sqrt{\lambda_2/\lambda_0 - (\lambda_1/\lambda_0)^2}}{\lambda_1/\lambda_0} = \sqrt{\frac{\lambda_0\lambda_2}{\lambda_1^2} - 1}
\end{aligned}
\tag{18}
$$

However, instead of the coefficient of variation it is common to use measures of bandwidth that take values only in the range 0 to 1. A definition that accomplishes this is (Lutes and Sarkani 1997)

$$\alpha_m = \frac{\lambda_m}{\sqrt{\lambda_0 \lambda_{2m}}} \tag{19}$$

which for m=1 corresponds to the above coefficient of variation because

$$\alpha_1 = \frac{1}{\sqrt{1+\delta^2}} \tag{20}$$

The bandwidth measures α_m approach zero for very broadband process, and approach unity for very narrowband processes. The most popular version of Eq. (19) is

$$\alpha_2 = \frac{\lambda_2}{\sqrt{\lambda_0 \lambda_4}} \tag{21}$$

This measure, and other ones, can be related to the variance of the process and its derivative, as described under derivative processes.

Design Spectra

In ship and offshore design, design spectra are employed to model the sea-surface elevation, η. An important spectrum, historically, is the Pierson-Moskowitz spectrum (P-M), which is a one-sided spectrum, i.e., $S^+(\omega)$, with the form

$$S_\eta(\omega) = \frac{\alpha \cdot g^2}{\omega^5} \cdot \exp\left(-\beta \cdot \left(\frac{\omega_0}{\omega} \right)^4 \right) \tag{22}$$

That spectrum gives rise to a family of design spectra, including P-M, ISSC, B-M, and ITTC. They share the spectrum shape

$$S_\eta(\omega) = \frac{A}{\omega^5} \cdot e^{-\frac{B}{\omega^4}} \tag{23}$$

where the parameters A and B are given in Table 1. Another design spectrum is Darbyshire-Scott, not yet written out here. Yet another option is the JONSWAP spectrum (from the Joint North Sea Wave Observation Project), which is essentially the Pierson-Moskowitz spectrum multiplied by a factor that accentuates the peak of the spectrum:

$$S_\eta(\omega) = \frac{\alpha \cdot g^2}{\omega^5} \cdot \exp\left(-\frac{5}{4} \cdot \left(\frac{\omega_p}{\omega} \right)^4 \right) \cdot \gamma^r \tag{24}$$

where

$$r = \exp\left(-\frac{(\omega - \omega_p)^2}{2 \cdot \sigma^2 \cdot \omega_p^2} \right) \tag{25}$$

Table 1: Parameters in the P-M, ISSC, B-M, and ITTC spectra.

	A	B
P-M	$\frac{\omega_z^4 \cdot H_s^2}{4 \cdot \pi}$	$\frac{\omega_z^4}{\pi}$
ISSC	$0.11 \cdot \omega_{01}^4 \cdot H_s^2$	$0.44 \cdot \omega_{01}^4$
B-M	$\frac{1}{2\pi} \cdot \left(0.257 \cdot H_{1/3}^2 \cdot T_{1/3} \cdot \left(\frac{2\pi}{T_{1/3}} \right)^5 \right)$	$1.03 \cdot \left(\frac{2\pi}{T_{1/3}} \right)^4$
ITTC	$8.1 \cdot 10^{-3} \cdot g^2$	$\frac{3.11}{H_s^2}$

Amplitude and Phase Spectra

To be written.

Derivative Processes

Time-derivatives of a stochastic process appear as physical quantities, such as velocity and acceleration, in differential equations, and also as auxiliary quantities in the study of response statistics. Consider the first-order derivative of the process $X(t)$:

$$\dot{X}(t) = \frac{dX(t)}{dt} \tag{26}$$

In a finite difference or Riemann sense, this derivative is equal to the limit

$$\dot{X}(t) = \lim_{h \to 0} \frac{X(t+h) - X(t)}{h} \tag{27}$$

For convenience, the argument in Eq. (27) is defined as

$$Y(t,h) \equiv \frac{X(t+h) - X(t)}{h} \tag{28}$$

As a result, the mean of the derivative process is

$$\begin{aligned} \mu_{\dot{X}}(t) &= \lim_{h \to 0} E\left[Y(t,h)\right] = \lim_{h \to 0} E\left[\frac{X(t+h) - X(t)}{h} \right] \\ &= \lim_{h \to 0} E\left[\frac{\mu_X(t+h) - \mu_X(t)}{h} \right] = \frac{d\mu_X(t)}{dt} \end{aligned} \tag{29}$$

which says that the "mean of the derivative is the derivative of the mean." Similarly, the cross-correlation function between $X(t)$ and $\dot{X}(t)$ is:

$$\begin{aligned}\phi_{X\dot{X}}(t_1,t_2) &= \lim_{h\to 0} E\left[X(t_1)\cdot Y(t_2,h)\right] = \lim_{h\to 0} E\left[X(t_1)\cdot\frac{X(t_2+h)-X(t_2)}{h}\right] \\ &= \lim_{h\to 0} E\left[\frac{\phi_{XX}(t_1,t_2+h)-\phi_{XX}(t_2)}{h}\right] = \frac{d\phi_{XX}(t_1,t_2)}{dt_2}\end{aligned} \tag{30}$$

and the autocorrelation function for $\dot{X}(t)$ is:

$$\begin{aligned}\phi_{\dot{X}\dot{X}}(t) &= \lim_{\substack{h_1\to 0\\ h_2\to 0}} E\left[Y(t_1,h_1)\cdot Y(t_2,h_2)\right] \\ &= \lim_{\substack{h_1\to 0\\ h_2\to 0}} E\left[\frac{X(t_1+h_1)-X(t_1)}{h_1}\cdot\frac{X(t_2+h_2)-X(t_2)}{h_2}\right] \\ &= \lim_{\substack{h_1\to 0\\ h_2\to 0}} E\left[\frac{X(t_1+h_1)X(t_2+h_2)-X(t_2+h_2)X(t_1)-X(t_1+h_1)X(t_2)+X(t_1)X(t_2)}{h_1h_2}\right] \\ &= \lim_{\substack{h_1\to 0\\ h_2\to 0}}\left(\frac{\phi_{XX}(t_1+h_1,t_2+h_2)-\phi_{XX}(t_2+h_2,t_1)-\phi_{XX}(t_1+h_1,t_2)+\phi_{XX}(t_1,t_2)}{h_1h_2}\right) \\ &= \frac{\partial^2\phi_{XX}(t_1,t_2)}{\partial t_1\,\partial t_2}\end{aligned} \tag{31}$$

where the last equality is the finite difference definition of a two-variable derivative. For second-moment stationary processes only one argument is necessary, denoted $\tau=|t_1-t_2|$. Adopting the earlier notation, this gives

$$R_{X\dot{X}}(\tau) = \frac{dR_{XX}(t_1-t_2)}{dt_2} = -\frac{dR_{XX}(\tau)}{d\tau} \tag{32}$$

$$R_{\dot{X}\dot{X}}(\tau) = \frac{\partial^2 R_{XX}(t_1-t_2)}{\partial t_1\,\partial t_2} = -\frac{\partial^2 R_{XX}(\tau)}{\partial\tau^2} \tag{33}$$

For second-moment stationary processes, $R_{XX}(\tau)$ is symmetric, which means its derivative at τ=0 vanishes. According to Eq. (32), this means that $R_{X\dot{X}}(0)=0$, which in turn means that $X(t)$ and $\dot{X}(t)$ are uncorrelated at any given time instant. Derivative processes can also be analyzed in the frequency domain. Assuming a solution on the form

$$X(t) = e^{i\omega t} \tag{34}$$

the derivative is

$$\dot{X}(t) = i\omega\cdot e^{i\omega t} \tag{35}$$

In other words, $i\omega$ is the transfer function between the "input" process X(t) and the "output" process $\dot{X}(t)$. As shown in the document on stochastic dynamics, where another system than "output is time-derivative of input" is considered, the output spectrum is the modulus of the transfer function, squared, times the input spectrum, which here means that:

$$S_{\dot{X}}(\omega) = \omega^2 \cdot S_X(\omega) \tag{36}$$

This is the reason why

$$\lambda_2 = \sigma_{\dot{X}}^2 \tag{37}$$

and

$$\lambda_4 = \sigma_{\ddot{X}}^2 \tag{38}$$

in addition to the fact that

$$\lambda_0 = \sigma_X^2 \tag{39}$$

Gaussian Processes

A process is Gaussian when the random variable $X(t)$ is normally distributed for all t. In the same way as the probability distribution of a Gaussian, i.e., normal random variable is fully described by the mean and standard deviation, a Gaussian stochastic process is fully described by the mean function and autocorrelation/autocovariance function. In addition to this major convenience, Gaussian stochastic processes are also popular because there is often insufficient data to justify another distribution type.

Generation of Realizations

One simple technique for creating realizations of a continuous stochastic process is to create it as a sum of trigonometric functions:

$$x(t) = \sum_{i=1}^{N} \left(A_i \cdot \cos(\omega_i t) + B_i \cdot \sin(\omega_i t) \right) \tag{40}$$

where A_i and B_i are random variables with properties to be determined shortly In preparation for this, the frequency axis of the PSD is discretized into N intervals of length $\Delta\omega$. The centre frequency in each interval is denoted ω_i. To determine the value of the coefficients A_i and B_i corresponding to that frequency, first consider the autocovariance function from Eq. (14) expressed in terms of the discretized one-sided PSD:

$$C_{XX}(\tau) = \int_0^{\infty} S_X^+(\omega) \cdot \cos(\omega\tau)\, d\omega = \sum_{i=1}^{N} \left(S_X^+(\omega_i) \cdot \Delta\omega \cdot \cos(\omega_i \tau) \right) \tag{41}$$

As shown in Eq. (15), the variance of the process equals the autocovariance function evaluated at τ=0, hence Eq. (41) reveals the variance at each frequency:

$$C_{XX}(0) = \sum_{i=1}^{N} \left(S_X^+(\omega_i) \cdot \Delta\omega \right) \tag{42}$$

In short, $S_X^+(\omega_i)\cdot\Delta\omega$ is the variance at the frequency ω_i. The objective now is to ensure that the generated process in Eq. (40) has that same variance at each frequency. Because Eq. (40) is a linear function of random variables its variance is

$$\text{Var}\big[x(t)\big]=\sum_{i=1}^{N}\Big(\text{Var}\big[A_i\big]\cdot\cos^2(\omega_i t)+\text{Var}\big[B_i\big]\cdot\sin^2(\omega_i t)\Big) \tag{43}$$

By selecting Var[A_i]=Var[B_i]=σ_i^2, where σ_i^2 is the variance at frequency number i, Eq. (43) yields

$$\begin{aligned}\text{Var}\big[x(t)\big]&=\sum_{i=1}^{N}\sigma_i^2\cdot\Big(\cos^2(\omega_i t)+\sin^2(\omega_i t)\Big)\\&=\sum_{i=1}^{N}\sigma_i^2\end{aligned} \tag{44}$$

Comparing Eq. (44) and Eq. (42) it is clear that the random variables A_i and B_i should be generated to have equal variances equal to $S_X^+(\omega_i)\cdot\Delta\omega$. Often their means are selected to be zero, to generate a zero-mean process, but a non-zero constant mean function equal to μ is obtained by generating random variables A_i and B_i with the same mean μ. The distribution type of the random variables is usually selected to be normal because given the linear form of Eq. (40) this implies that the amplitude of the process at any time instant is also normal, i.e., it is a Gaussian process.

Another approach for generating realizations of a continuous stochastic process is

$$x(t)=c_0+\sum_{i=1}^{\infty}\big(c_i\cdot\sin(\omega_i t+\phi_i\big) \tag{45}$$

where c_i is the amplitude and ϕ_i is the phase angle associated with each frequency ω_i. This formulation requires the amplitude spectrum and the phase spectrum, which will be described in an upcoming version of this document.

Statistical Inference

Given an observed realization of an ergodic stochastic process, an estimate of its mean is:

$$\mu_X=\lim_{N\to\infty}\frac{1}{T}\cdot\int_{T/2}^{T/2}x(t)\,dt=\lim_{N\to\infty}\frac{1}{N}\cdot\sum_{i=1}^{N}x(t_i) \tag{46}$$

The estimate of the autocorrelation function is

$$C_{XX}(\tau)=\lim_{T\to\infty}\frac{1}{T}\cdot\int_{T/2}^{T/2}x(t)\cdot x(t+\tau)\,dt=\lim_{N\to\infty}\frac{1}{N}\cdot\sum_{i=1}^{N}\big(x(t_i)\cdot x(t_i+\tau)\big) \tag{47}$$

References

Lutes, L. D., and Sarkani, S. (1997). *Stochastic analysis of structural and mechanical vibrations*. Prentice Hall.