Computational Cross-section Analysis

Structural analysis can be conducted "by hand" or using computers. The stiffness method and its extension, the finite element method, are examples of computational structural analysis methods. Computational methods can also be applied to cross-section analysis and that is the subject of this document. However, the application of the finite element method to cross-section analysis is addressed in other documents, such as the use of T3 elements for torsion analysis.

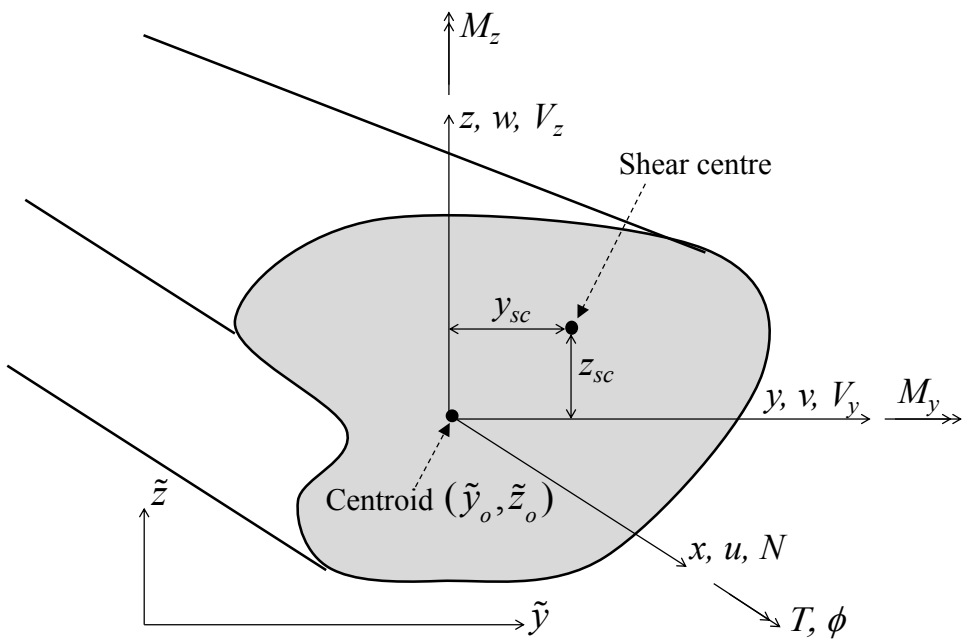

Figure 1: Notation for axes, deformations, and stress resultants.

In general, cross-section analysis has two objectives: to determine stresses and to determine cross-section constants. For now, only the latter is addressed in this document. Some of the relevant notation is given in Figure 1, including the coordinates of the centroid, $(\tilde{y}_o, \tilde{z}_o)$, and the coordinates of the shear centre, (y_{sc}, z_{sc}). The other cross-section constants addressed in this document are

- A = area
- I_y = moment of inertia about the y-axis
- I_z = moment of inertia about the z-axis
- I_{yz} = product of inertia
- A_{sy} = shear area for shear force V_y
- A_{sz} = shear area for shear force V_z
- J = cross-section constant for St. Venant torsion
- C_w = cross-section constant for warping torsion

Thin-walled Cross-sections

The right-hand side of Figure 2 shows one part, A-B, of an open thin-walled cross-section. A generic thin-walled cross-section contains n straight parts. The thickness of each part is constant and equal to t_i, where i=1,2,…,n.

Area

The cross-sectional area is:

$$A = \sum_{i=1}^{n} t_i \cdot L_i \tag{1}$$

where the length of each part is

$$L_i = \sum_{i=1}^{n} \sqrt{\Delta y_i^2 + \Delta z_i^2} \tag{2}$$

where

$$\Delta y_i = \tilde{y}_{Bi} - \tilde{y}_{Ai} \quad \text{and} \quad \Delta z_i = \tilde{z}_{Bi} - \tilde{z}_{Ai} \tag{3}$$

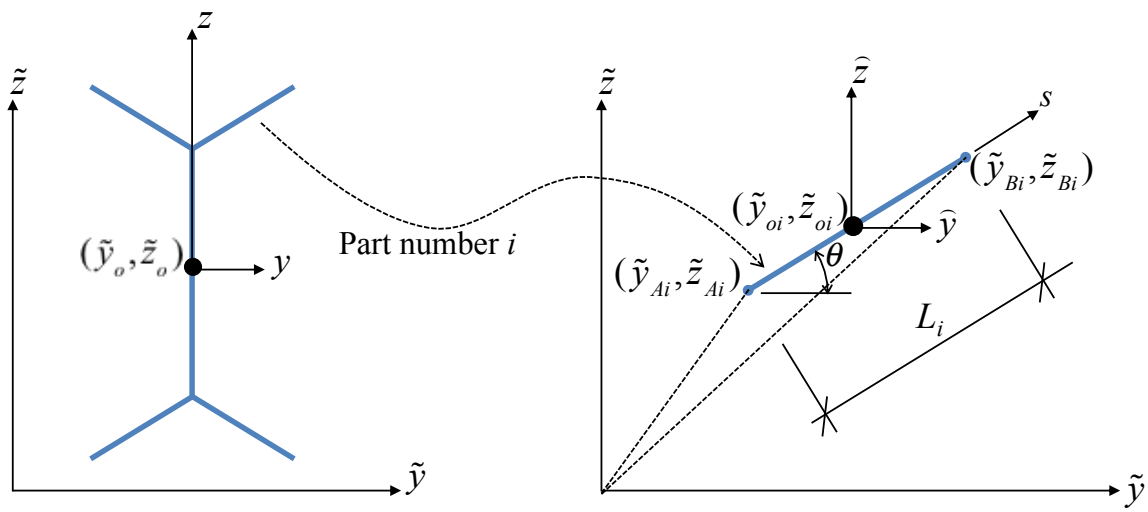

Figure 2: One part of an open thin-walled cross-section, three coordinate systems.

Centroid

The coordinates of the centroid are

$$\tilde{y}_o = \frac{\sum_{i=1}^{n} \tilde{y}_{oi} A_i}{A} \tag{4}$$

and

$$\tilde{z}_o = \frac{\sum_{i=1}^{n} \tilde{z}_{oi} A_i}{A} \tag{5}$$

Moments of Inertia

For each cross-section part, the contribution to the moment of inertia about the *y*-axis is

$$I_y = \sum_{i=1}^{n} I_{oyi} + z_{oi}^2 \cdot A_i = \sum_{i=1}^{n} I_{oyi} + (\tilde{z}_{oi} - \tilde{z}_o)^2 \cdot A_i \tag{6}$$

where

$$I_{oyi} = \int_{A_i} \hat{z}^2 \, dA = 2 \cdot t_i \cdot \int_0^{L_i/2} \hat{z}^2 \, ds = 2 \cdot t_i \cdot \int_0^{L_i/2} \left(s \cdot \sin(\theta_i) \right)^2 ds = \frac{t_i \cdot L_i^2}{12} \sin^2(\theta_i) \tag{7}$$

where *z* in Eq. (7) runs from the local centroid of Part *i* of the cross-section. Similarly,

$$I_z = \sum_{i=1}^{n} I_{ozi} + y_{oi}^2 \cdot A_i = \sum_{i=1}^{n} I_{ozi} + (\tilde{y}_{oi} - \tilde{y}_o)^2 \cdot A_i \tag{8}$$

and

$$I_{ozi} = \frac{t_i \cdot L_i^3}{12} \cos^2(\theta_i) \tag{9}$$

Product of Inertia

Following the notation used for moments of inertia, the product of inertia is

$$I_{yz} = \sum_{i=1}^{n} I_{oyzi} + y_{oi} \cdot z_{oi} \cdot A_i = \sum_{i=1}^{n} I_{oyzi} + (\tilde{y}_{oi} - \tilde{y}_o) \cdot (\tilde{z}_{oi} - \tilde{z}_o) \cdot A_i \tag{10}$$

where

$$\begin{aligned} I_{oyzi} &= \int_{A_i} \hat{y} \cdot \hat{z} \, dA \\ &= 2 \cdot t_i \cdot \int_0^{L_i/2} \hat{y} \cdot \hat{z} \, ds \\ &= 2 \cdot t_i \cdot \int_0^{L_i/2} \left(s \cdot \cos(\theta_i) \right) \left(s \cdot \sin(\theta_i) \right) ds \\ &= \frac{t_i \cdot L_i^3}{12} \cdot \sin(\theta_i) \cdot \cos(\theta_i) \end{aligned} \tag{11}$$

and the observation is made that I_{oyzi}=0 for θ=0° and θ=90°.

St. Venant Torsion Constant

For thin-walled open cross-sections the calculation is simple:

$$J = \frac{1}{3} \cdot \sum_{i=1}^{n} t_i^3 \cdot L_i \tag{12}$$

For closed cross-sections the algorithm is a matter of calculating the area of cells and picking up contributions to the integral of 1/*t* around the cell:

$$J = \frac{4 \cdot A_m^2}{\oint_{\Gamma_m} \frac{1}{t} \mathrm{d}s} \tag{13}$$

Cross-products can be utilized to calculate the area, which is a key topic next, when warping is addressed.

Warping

A central quantity in the analysis of cross-sections is the "omega diagram," $\Omega(y,z)$, the warping caused by a unit rate of twist. The omega diagram facilitates the determination of the shear centre location and the warping torsion constant C_w. The computational procedure starts with the calculation of a "trial" omega diagram. The reference point for that trial diagram is here taken to be the origin of the coordinate system used in the input, i.e., the tilde system shown in Figure 2. The omega diagram is defined by

$$\Omega(s) = \int \left(h - \frac{J}{2 \cdot t \cdot A_m} \right) ds = \int \left(h - \bar{h} \right) ds \tag{14}$$

where h is the distance from the trial point to the tangent of the cross-section part. When $\bar{h} = 0$ then the contribution from each cross-section part to Ω is given as $\Delta\Omega$ in Figure 3. When $\bar{h} \neq 0$ then

$$\Delta\Omega = \tilde{y}_{Bi} \cdot \tilde{z}_{Ai} - \tilde{y}_{Ai} \cdot \tilde{z}_{Bi} - \frac{J \cdot L_i}{2 \cdot t_i \cdot A_m} \tag{15}$$

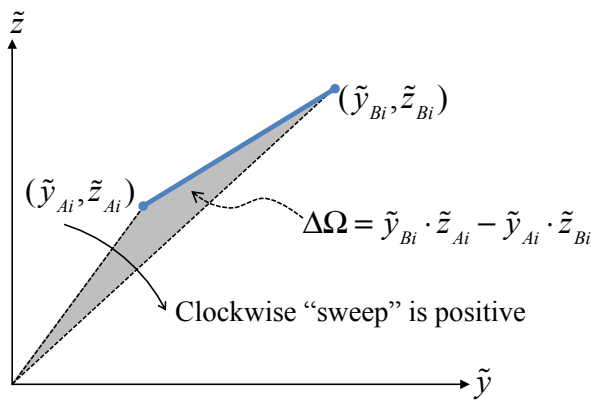

Figure 3: Using cross-products to establish the omega diagram.

The contributions to the cross product from different parts of the cross-section are positive or negative. Informally speaking, when a part is swiped by an imaginary clockwise radar sweep emanating from the trial point then the contribution is positive. For that reason, the cross-product of the vectors (y_{Bi},z_{Bi}) and (y_{Ai},z_{Ai}), both in sign and value, is the appropriate contribution to the omega diagram over one part:

$$\Omega_{trial} = \sum_{i=1}^{n} \tilde{y}_{Bi} \tilde{z}_{Ai} - \tilde{y}_{Ai} \tilde{z}_{Bi} - \frac{J}{2 \cdot t_i \cdot A_m} \tag{16}$$

where the last term cancels for open cross-sections. To obtain the final omega diagram it is necessary to evaluate the following integrals:

$$\int_A \Omega_{trial}\, dA, \quad \int_A y \cdot \Omega_{trial}\, dA, \quad \int_A z \cdot \Omega_{trial}\, dA \tag{17}$$

As explained in the document on warping torsion the final omega diagram is then

$$\Omega_{final} = \Omega_{trial} - \frac{\int_A \Omega_{trial}\, dA}{A} - \frac{\int_A y \cdot \Omega_{trial}\, dA}{I_z} \cdot y - \frac{\int_A z \cdot \Omega_{trial}\, dA}{I_y} \cdot z \tag{18}$$

Denoting the trial omega values at Point A and Point B in Figure 2 as Ω_A and Ω_B, respectively, the integrals in Eq. (17) are

$$\int_A \Omega_{trial}\, dA = \sum_{i=1}^{n} \frac{\Omega_{Ai} + \Omega_{Bi}}{2} \cdot L_i \cdot t_i \tag{19}$$

and

$$\begin{aligned} \int_A y \cdot \Omega_{trial}\, dA &= \sum_{i=1}^{n} \int_0^{L_i} \left(y_{Ai} + \frac{s}{L_i}(y_{Bi} - y_{Ai}) \right) \cdot \left(\Omega_{Ai} + \frac{s}{L_i}(\Omega_{Bi} - \Omega_{Ai}) \right) \cdot t_i\, ds \\ &= \sum_{i=1}^{n} \frac{L_i t_i}{6} \left(y_{Ai} \left(2 \cdot \Omega_{Ai} + \Omega_{Bi} \right) + y_{Bi} \left(2 \cdot \Omega_{Bi} + \Omega_{Ai} \right) \right) \end{aligned} \tag{20}$$

and similarly,

$$\int_A z \cdot \Omega_{trial}\, dA = \sum_{i=1}^{n} \frac{L_i t_i}{6} \left(z_{Ai} \left(2 \cdot \Omega_{Ai} + \Omega_{Bi} \right) + z_{Bi} \left(2 \cdot \Omega_{Bi} + \Omega_{Ai} \right) \right) \tag{21}$$

where it is emphasized that y_{Ai}, y_{Bi}, z_{Ai}, and z_{Bi} are measured in the coordinate system originating at the centroid of the entire cross-section.

Shear Centre

Eqs. (20) and (21) in the determination of the warping diagram above implicitly determine the shear centre coordinates because

$$\tilde{y}_{sc} = -\frac{\int_A z \cdot \Omega_{trial}\, dA}{I_y} \tag{22}$$

and

$$\tilde{z}_{sc} = \frac{\int_A y \cdot \Omega_{trial}\, dA}{I_z} \cdot y \tag{23}$$

Warping Torsion Constant

The cross-section constant for warping torsion is obtained by integrating the squared of the final omega diagram:

$$C_\omega = \int_A \Omega^2 \, dA = \sum_{i=1}^{n} \frac{L_i t_i}{3} \left(\Omega_{Ai}^2 + \Omega_{Bi}^2 \pm \Omega_{Ai} \Omega_{Bi} \right) \tag{24}$$

where the sign in question is positive if Ω_{Ai} has the same sign as Ω_{Bi}.

Shear Area

The document on Timoshenko beam theory defines the shear area as $A_v = \beta A$ with

$$\beta = \frac{I^2}{A \cdot \int_A \left(\frac{Q(s)}{t} \right)^2 dA} \tag{25}$$

where $Q(s)$ is the first moment-of-area for the area A_s, which is the area that is outside a "cut" made at the location s somewhere along the cross-section part:

$$Q(s) = \int_{A_s} z \, dA = \int_{A_s} z \cdot t \, ds \quad \text{for shear force in the } z\text{-direction, and} \tag{26}$$

$$Q(s) = \int_{A_s} y \cdot t \, ds \quad \text{for shear force in the } y\text{-direction.} \tag{27}$$

The aforementioned cut must be made at all possible locations along the cross-section in order to accurately evaluate the integral in the denominator of Eq. (25). For a given cut location, denoted by s, the sought first moment of area is

$$\begin{aligned} Q(s) &= \int_{A_s} z(s) \cdot t \, ds \\ &= \underbrace{\left(z_o \cdot L \cdot t \right)}_{\substack{\text{For part that} \\ \text{crosses cut}}} + \sum_{i=1}^{n_{As}} z_{oi} \cdot A_i \\ &= \underbrace{\left(\frac{\tilde{z}_* + \tilde{z}_B}{2} \cdot \sqrt{\left(\tilde{y}_B - \tilde{y}_* \right)^2 + \left(\tilde{z}_B - \tilde{z}_* \right)^2} \cdot t \right)}_{\text{For part that crosses cut}} + \sum_{i=1}^{n_{As}} \left(\tilde{z}_{oi} - \tilde{z}_o \right) \cdot A_i \end{aligned} \tag{28}$$

where n_{As} is the number of parts outside the cut and

$$\begin{aligned} \tilde{y}_{*i} &= \tilde{y}_{Ai} + s \cdot \frac{\tilde{y}_{Bi} - \tilde{y}_{Ai}}{L_i} \\ \tilde{z}_{*i} &= \tilde{z}_{Ai} + s \cdot \frac{\tilde{z}_{Bi} - \tilde{z}_{Ai}}{L_i} \end{aligned} \tag{29}$$

where s runs from x>0 to L, where L=length of the part that crosses the cut. The value of x varies as the location of the cut varies along the part. $Q(s)$ for shear force in the y-direction is obtained similarly.

Solid Cross-sections

Computational analysis for a solid cross-section requires a finite element mesh over the entire cross-section. In this document a mesh of n triangular elements is assumed. An example of such a mesh is shown in Figure 1. In the following it is assumed that the coordinates of the three nodes of each element is known, and that they are numbered counter-clockwise, as in Figure 1.

Area

The area of the cross-section can be calculated as a cross-product, which has the added benefit of allowing a check of whether the mesh has elements that are numbered clockwise. For this purpose, the vector from Node 1 to Node 2 labelled $\mathbf{v}_{12}$ and the vector from Node 1 to Node 3 labelled $\mathbf{v}_{13}$. For each element the cross-product is

$$\mathbf{v}_{12} \times \mathbf{v}_{13} = (\tilde{y}_2 - \tilde{y}_1)(\tilde{z}_3 - \tilde{z}_1) - (\tilde{y}_3 - \tilde{y}_1)(\tilde{z}_2 - \tilde{z}_1) \tag{30}$$

If that cross-product comes out negative then an error has been made in the numbering of the nodes. The total area of the cross-section is

$$A = \sum_{i=1}^{n} A_i = \sum_{i=1}^{n} \frac{(\mathbf{v}_{12} \times \mathbf{v}_{13})_i}{2} \tag{31}$$

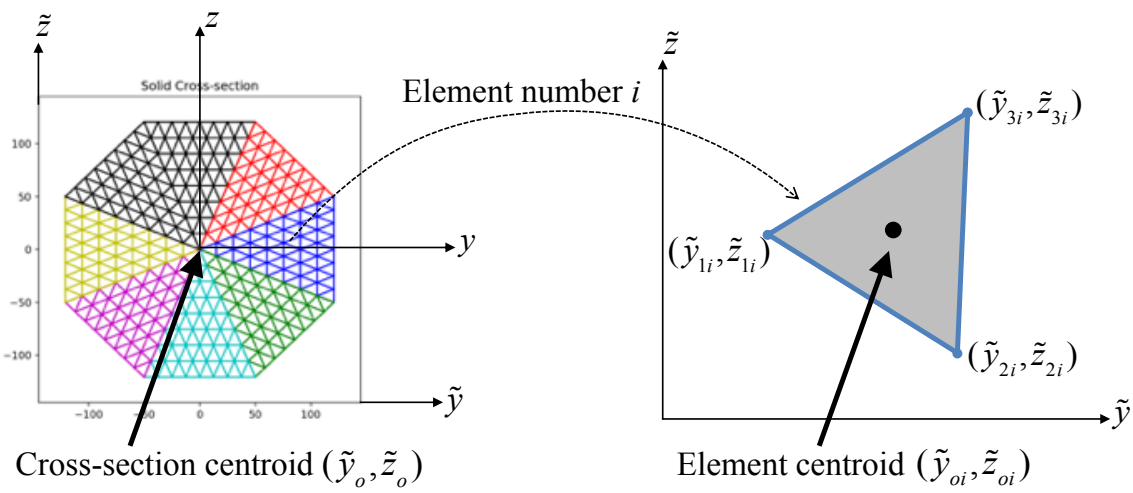

Figure 4: Mesh for octagonal solid cross-section generated with Python.

Centroid

The coordinates of the centroid are given by Eqs. (4) and (5), with the element areas given by Eq. (31) and the local centroid coordinates given by

$$\tilde{y}_{oi} = \frac{(\tilde{y}_1 + \tilde{y}_2 + \tilde{y}_3)_i}{3} \tag{32}$$

and

$$\tilde{z}_{oi} = \frac{(\tilde{z}_1 + \tilde{z}_2 + \tilde{z}_3)_i}{3} \tag{33}$$

Moments of Inertia

At first glance, the centroid coordinates are needed in the calculation of the moments of inertia. However, it is unnecessary to "loop through" the triangles twice, first to calculate the centroid coordinates, and second to calculate the moments of inertia. That is because

$$\begin{aligned} I_y &= \int_A z^2 \, dA \\ &= \int_A (\tilde{z} - z_o)^2 \, dA \\ &= \int_A (\tilde{z}^2 - 2\tilde{z} z_o + \tilde{z}_o^2) \, dA \\ &= \int_A \tilde{z}^2 \, dA - 2 z_o \int_A \tilde{z} \, dA + \tilde{z}_o^2 A \end{aligned} \tag{34}$$

As a result,

$$I_y = \left(\sum_{i=1}^{n} \tilde{z}_{oi}^2 A_i \right) - 2 \cdot \tilde{z}_o \cdot \left(\sum_{i=1}^{n} \tilde{z}_{oi} A_i \right) + \tilde{z}_o^2 A \tag{35}$$

Similarly

$$I_z = \left(\sum_{i=1}^{n} \tilde{y}_{oi}^2 A_i \right) - 2 \cdot \tilde{y}_o \cdot \left(\sum_{i=1}^{n} \tilde{y}_{oi} A_i \right) + \tilde{y}_o^2 A \tag{36}$$

Product of Inertia

Following the approach adopted for the moments of inertia, the product of inertia is

$$\begin{aligned} I_{yz} &= \int_A y \cdot z \, dA \\ &= \int_A (\tilde{y} - y_o)(\tilde{z} - z_o) \, dA \\ &= \int_A (\tilde{y} \cdot \tilde{z} + y_o \cdot z_o - \tilde{y} \cdot z_o - \tilde{z} \cdot y_o) \, dA \\ &= \int_A \tilde{y} \cdot \tilde{z} \, dA + y_o \cdot z_o \cdot A - z_o \cdot \int_A \tilde{y} \, dA - y_o \cdot \int_A \tilde{z} \, dA \end{aligned} \tag{37}$$

where the last two integrals are addressed by Eqs. (35) and (36) and the first integral is

$$\int_A \tilde{y} \cdot \tilde{z} \, dA = \sum_{i=1}^{n} \tilde{y}_{oi} \tilde{z}_{oi} A_i \tag{38}$$

Shear Areas

The shear area of a cross-section is defined as $A_v = \beta A$, where A is the total cross-section area and

$$\beta = \frac{I^2}{A \cdot \int_A \left(\frac{Q}{t} \right)^2 dA} \tag{39}$$

where

$$Q = \int_{A_s} z\,dA = \int_{A_s} (\tilde{z} - z_o)\,dA = \int_{A_s} \tilde{z}\,dA - \tilde{z}_o \cdot A = \sum_{i=1}^{n} \tilde{z}_{oi} A_i - \tilde{z}_o \cdot A \tag{40}$$

and caution must be applied in the specification of the boundaries of A_s in the code.

Finite Element Analysis for the Rest

There are several items outstanding in this section on solid cross-section, including the calculation of y_{sc}, z_{sc}, J and C_w. Those items are addressed in another documents that applies the finite element method to cross-section analysis. Triangular "T3 elements" is one option in that regard. The determination of the warping of the cross-section due to a unit rate of twist, i.e., the omega function, is central such finite element analyses. Meshing the cross-section with finite elements can also yield improved estimates of the moments of inertia; more on that in the document on finite element analysis of cross-sections.

References

Bell, K. (2013). *An engineering approach to finite element analysis of linear structural mechanics problems*. Fagbokforlaget, Norway.

Meek, J. L. (1991). *Computer Methods in Structural Analysis*. E & FN Spon.