Timoshenko Beams

The Euler-Bernoulli beam theory neglects shear deformations by assuming that plane sections remain plane and perpendicular to the neutral axis during bending. As a result, shear strains and stresses are removed from the theory. Shear forces are only recovered later by equilibrium: $V=dM/dx$. In reality, the beam cross-section deforms somewhat like what is shown in Figure 1c. This is particularly the case for deep beams, i.e., those with relatively high cross-sections compared with the beam length, when they are subjected to significant shear forces. Usually the shear stresses are highest around the neutral axis, which is where; consequently, the largest shear deformation takes place. Hence, the actual cross-section curves. Instead of modelling this curved shape of the cross-section, the Timoshenko beam theory retains the assumption that the cross-section remains plane during bending. However, the assumption that it must remain perpendicular to the neutral axis is relaxed. In other words, the Timoshenko beam theory is based on the shear deformation mode in Figure 1d.

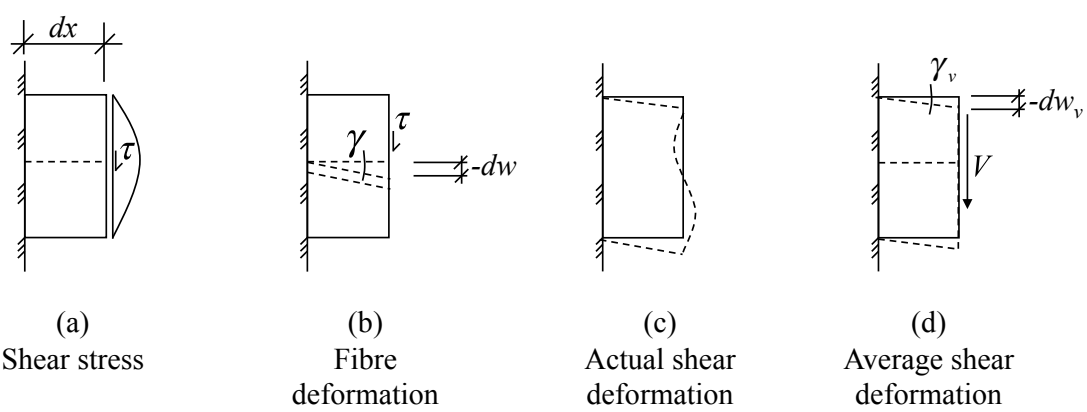


Figure 1: Shear deformation.

The average shear deformation in Figure 1d is linked with reality, i.e., the shear deformation in Figure 1b and Figure 1c by equality of work. It is required that the work carried out in the average deformation must equal the sum of the work carried out by all fibres deforming:

$$\int_A \frac{1}{2} \cdot dw \cdot \tau \; dA = \frac{1}{2} \cdot dw_v \cdot V \tag{1}$$

Substitution of the kinematic relationship $w=\gamma\, dx$ yields:

$$\int_A \frac{1}{2} \cdot (\gamma \cdot dx) \cdot \tau \; dA = \frac{1}{2} \cdot (\gamma_v \cdot dx) \cdot V \tag{2}$$

Substitution of the material law $\tau=G\gamma$, where G is the shear modulus defined by $G=E/(2(1+\nu))$, yields:

$$\int_A \frac{1}{2} \cdot \left(\frac{\tau}{G} \right) \cdot dx \cdot \tau \; dA = \frac{1}{2} \cdot \left(\frac{\tau_v}{G} \right) \cdot dx \cdot V \tag{3}$$

Caution must be applied in the interpretation of τ_v. It is NOT simply the average shear stress obtained by smearing the shear force, V, uniformly over the entire cross-section area. If that mistake was made then the right-hand side of Eq. (3) would NOT match the left-hand side. Instead, the shear force must be uniformly smearing over a precise fraction of the total area in order for Eq. (3) to be valid. That precise area is called the "shear area," A_v, so that $\tau_v = V/A_v$ and consequently

$$\int_A \frac{1}{2} \cdot \left(\frac{\tau}{G} \right) \cdot dx \cdot \tau \; dA = \frac{1}{2} \cdot \left(\frac{V}{A_v} \right) \cdot \frac{1}{G} \cdot dx \cdot V \tag{4}$$

Substitution of the expression for shear stress from Euler-Bernoulli beam theory on the left-hand side, and definition of the shear area as $A_v = \beta A$, where β is a constant that is defined shortly yields:

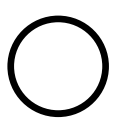
$$\int_A \frac{1}{2} \cdot \frac{1}{G} \cdot dx \cdot \left(\frac{V \cdot Q}{I \cdot t} \right)^2 dA = \frac{1}{2} \cdot \frac{V}{\beta \cdot A} \cdot \frac{1}{G} \cdot dx \cdot V \tag{5}$$

Solving Eq. (5) for β yields:

$$\beta = \frac{I^2}{A \cdot \int_A \left(\frac{Q}{t} \right)^2 dA} \tag{6}$$

The constant β tells how much of the cross-section that the shear force is "smeared" uniformly over. The value of β for a few typical cross-sections is provided in Table 1.

Table 1: Value of β for some cross-section shapes.

Shape	β
	5/6
	9/10
	1/2
	$\sim A_{web}/A$

Upon determining the shear area, A_v, the shear deformation is readily included in the structural analysis, for example in the unit virtual load method:

$$\delta W_{ext} = \delta W_{int}$$
$$\Downarrow$$
$$\text{Virtual force} \cdot \text{Real displacement} = \int \text{Virtual force} \cdot \text{Real displacement} \tag{7}$$
$$\Downarrow$$
$$\delta F \cdot \Delta = \sum_{\substack{\text{Sum over}\\ \text{all members}}} \left(\underbrace{\delta N \cdot \frac{N \cdot L}{EA}}_{\text{Axial}} + \underbrace{\int_0^L \delta M \cdot \frac{M}{EI}\, \mathrm{d}x}_{\text{Flexure}} + \underbrace{\int_0^L \delta V \cdot \frac{V}{G \cdot A_v}\, \mathrm{d}x}_{\text{Shear}} \right)$$

Notice that the "real shear displacement" in Eq. (7) is the shear angle defined earlier:

$$\gamma_v = \frac{\tau_v}{G} = \frac{V}{GA_v} \tag{8}$$

In summary, Timoshenko beam theory implies that cross-sections remain place, but the rotation of the cross-section is no longer equal to the rotation of the beam axis. This is visualized in Figure 2, where it is observed that the total cross-section rotation has a flexural term and a shear term:

$$\theta = \frac{dw}{dx} + \gamma_v \tag{9}$$

with γ_v given in Eq. (8).

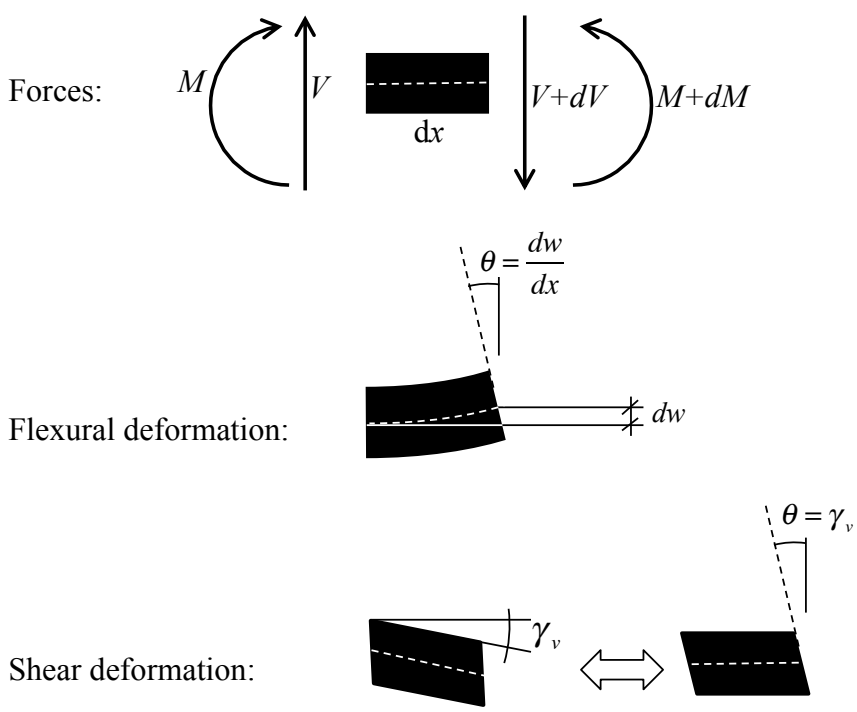


Figure 2: Two contributions to cross-section rotation.