3D Elasticity Theory

According to modern structural engineering structures break when the stress, or a stress resultant, exceed a limiting value. In other words, stress is today the key design criterion. In addition, the strain, or resulting deformation, is often considered in order to avoid uncomfortably large deflections. It is interesting to note that the beam theory we today name after Leonard Euler (1707-1783) and Daniel Bernoulli (1700-1782), and to some extent Daniel's uncle Jacob Bernoulli (1654-1705), came about before the concepts of stress and strain existed. While Euler and Bernoulli established the beam theory with the correct neutral axis it was Augustin Cauchy (1789-1857) who first introduced the concepts of stress and strain around 1822. Although Antoine Parent (1666-1716) and later Charles-Augustin de Coulomb (1736-1806) used similar concepts in beam theory, earlier work essentially considered the molecular forces between individual particles rather than stress distributed in a solid continuum. Cauchy took the continuum idea from hydrodynamics (Timoshenko 1953) and the concepts stress and strain in solids are now omnipresent in solid mechanics and structural engineering.

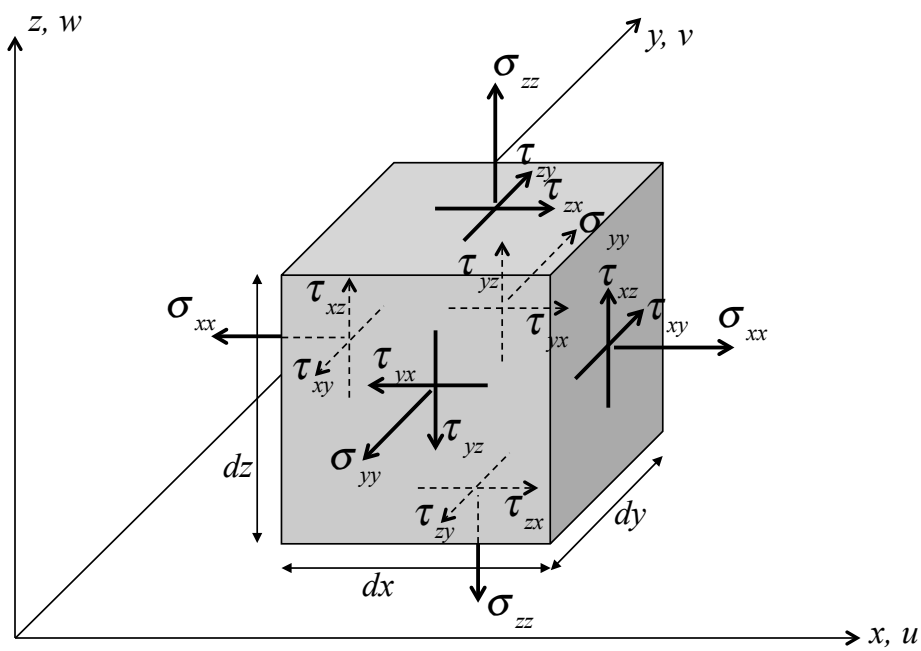

Figure 1: Stress components.

Figure 1 shows the stress components for an infinitesimally small cube of a continuum material. All stresses act in a coordinate direction hence these are called coordinate stresses. The first index indicates the direction of the normal vector to the plane where the stress acts and the second index indicates the direction of the stress. Axial stresses are positive in tension and shear stresses are positive when they act in the positive axis direction on a surface that has a positive axis direction as the surface normal. In Figure 1 σ=axial stress and τ=shear stress and when σ is used for both components with equal

indices are understood to be axial stresses and components with different indices are shear stresses. The coordinate stresses are collected in the stress tensor

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix} \equiv \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} \tag{1}$$

written σ_{ij} in index notation. Similarly the strain tensor is

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix} = \begin{bmatrix} \varepsilon_{xx} & 0.5\cdot\gamma_{xy} & 0.5\cdot\gamma_{xz} \\ 0.5\cdot\gamma_{yx} & \varepsilon_{yy} & 0.5\cdot\gamma_{yz} \\ 0.5\cdot\gamma_{zx} & 0.5\cdot\gamma_{zy} & \varepsilon_{zz} \end{bmatrix} \tag{2}$$

written ε_{ij} in index notation. Sometimes the stress and strain tensors are written in "Voight notation:"

$$\boldsymbol{\sigma} = \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix} \qquad \boldsymbol{\varepsilon} = \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} \tag{3}$$

Equilibrium

Here we are not concerned with externally applied loads or surface forces but rather equilibrium between the forces within a material particle. This includes stresses and potential forces acting on the volume, such as gravity. Figure 1 shows the stresses acting on an infinitesimally small material volume. The figure is hard to read because it includes the changes in stress-values from one end of the cube to the opposite side. For example, the axial stress σ_{xx} changes by $d\sigma_{xx}$ from one side to the other. The volume forces are not included in the figure but are denoted f_x, f_y, and f_z with index indicating the direction of the force. Equilibrium in the x-direction yields

$$d\sigma_{xx}\cdot dy\cdot dz + d\sigma_{yx}\cdot dx\cdot dz + d\sigma_{zx}\cdot dx\cdot dy + f_x\cdot dx\cdot dy\cdot dz = 0 \tag{4}$$

Dividing through by ($dx\ dy\ dz$) yields:

$$\frac{d\sigma_{xx}}{dx} + \frac{d\sigma_{yx}}{dy} + \frac{d\sigma_{zx}}{dz} + f_x = 0 \tag{5}$$

Repeating the exercise in all three axis-directions produces the equilibrium equations that all material particles that are in equilibrium must satisfy:

$$\sigma_{ij,i} + f_j = 0 \tag{6}$$

Moment equilibrium of the infinitesimal cube yields the symmetry of the stress tensor:

$$\sigma_{ij} = \sigma_{ji} \tag{7}$$

Kinematic Compatibility

Equations are here sought to relate strains with displacements. As shown in Figure 1 the displacements in the axis directions *x, y, z* are *u v w*, respectively. Expressions for longitudinal strains are obtained by studying an infinitesimal material cube. First consider the *x*-direction. The *x*-direction displacement at *x* is *u*. The *x*-direction displacement at *x*+d*x* is *u*+(∂*u*/∂*x*)d*x*. Defining strain as change in length divided by original length yields

$$\varepsilon_{xx} = \frac{\left(\frac{\partial u}{\partial x}\right) \cdot dx}{dx} = \frac{\partial u}{\partial x} \tag{8}$$

Repeating the consideration for the other axis directions yields

$$\begin{aligned} \varepsilon_{xx} &= \frac{\partial u}{\partial x} \\ \varepsilon_{yy} &= \frac{\partial v}{\partial y} \\ \varepsilon_{zz} &= \frac{\partial w}{\partial z} \end{aligned} \tag{9}$$

Now to the shear strains, starting with γ_{xy} visualized in Figure 2 and defining the change in angle between originally orthogonal lines. The change in angle has two contributions:

$$\gamma_{xy} = \gamma_{xy} = \varepsilon_{xy} + \varepsilon_{yx} = \frac{\left(\frac{\partial v}{\partial x}\right) \cdot dx}{dx} + \frac{\left(\frac{\partial u}{\partial y}\right) \cdot dy}{dy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \tag{10}$$

Here it is understood that the engineering shear strain γ_{xy} is twice the corresponding coordinate strains. Repeating the consideration for the other two coordinate planes yields

$$\begin{aligned} \gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \\ \gamma_{yz} &= \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \\ \gamma_{xz} &= \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \end{aligned} \tag{11}$$

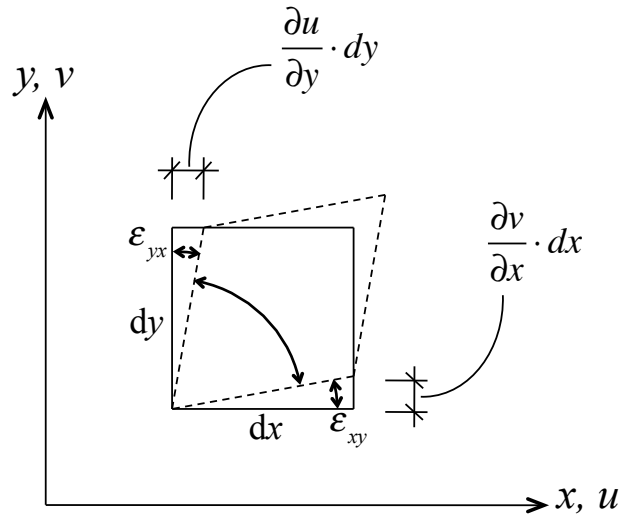

Figure 2: Shear strains.

The kinematic compatibility equations for both axial and shear strains are summarized by

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \tag{12}$$

By means of Voight notation the kinematic equations can be written in vector notation, with ∇ defined as a matrix differential operator:

$$\boldsymbol{\varepsilon} = \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} = \begin{bmatrix} \partial/\partial x & 0 & 0 \\ 0 & \partial/\partial y & 0 \\ 0 & 0 & \partial/\partial z \\ \partial/\partial y & \partial/\partial x & 0 \\ 0 & \partial/\partial z & \partial/\partial y \\ \partial/\partial z & 0 & \partial/\partial x \end{bmatrix} \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \begin{bmatrix} u_{,x} \\ v_{,y} \\ w_{,z} \\ u_{,y} + v_{,x} \\ v_{,z} + w_{,y} \\ u_{,z} + w_{,x} \end{bmatrix} = \nabla \mathbf{u} \tag{13}$$

Because the compatibility equations contain six strain components and only three displacement components, only certain strain patterns are physically possible. For that reason the strains-displacement equations are sometimes combined into "compatibility equations" that give conditions for valid deformation patterns. This is done below for 2D elasticity theory. Sometimes the terms compatibility equation and compatibility condition are maintained even when adding material law and equilibrium equations. Such equations are, together with boundary conditions, sufficient to determine the solution to specific problems.

Material Law

The theory of elasticity is founded on the assumption of a homogeneous isotropic linear elastic material. For a material particle, the relationship between a uniaxial stress and the

corresponding uniaxial strain is give by the modulus of elasticity, sometimes called Young's modulus, E, formulated in Hooke's law:

$$\sigma = E \cdot \varepsilon \tag{14}$$

The strain in the transversal direction is defined by Poisson's ratio, ν:

$$\nu \equiv \frac{\varepsilon_t}{\varepsilon} \qquad \Rightarrow \qquad \varepsilon_t = \nu \cdot \varepsilon \qquad \Rightarrow \qquad \varepsilon_t = -\nu \cdot \frac{\sigma}{E} \tag{15}$$

where ε_t is the transversal strain. Strain expressions that account for transversal strains in the orthogonal directions yield the three-dimensional version of Hooke's law:

$$\begin{aligned} \varepsilon_{xx} &= \frac{\sigma_{xx}}{E} - \nu \cdot \frac{\sigma_{yy}}{E} - \nu \cdot \frac{\sigma_{zz}}{E} \\ \varepsilon_{yy} &= \frac{\sigma_{yy}}{E} - \nu \cdot \frac{\sigma_{xx}}{E} - \nu \cdot \frac{\sigma_{zz}}{E} \\ \varepsilon_{zz} &= \frac{\sigma_{zz}}{E} - \nu \cdot \frac{\sigma_{xx}}{E} - \nu \cdot \frac{\sigma_{yy}}{E} \end{aligned} \tag{16}$$

There are only two independent parameters in the general Hooke's law. However, a special material constant named the shear modulus, G, which is related to E and ν, defines the relationship between shear stresses and shear strains:

$$\tau_{ij} = G \cdot \gamma_{ij} \quad , \quad i \neq j \tag{17}$$

To determine the relationship between G, E and ν, consider an infinitesimally small two-dimensional material particle subjected to pure shear τ. Mohr's circle for this case is centred at the origin with radius τ. Consequently, the principal stresses are $-\tau$ and τ with axes at 45°. The deformation of the particle is shown in Figure 3.

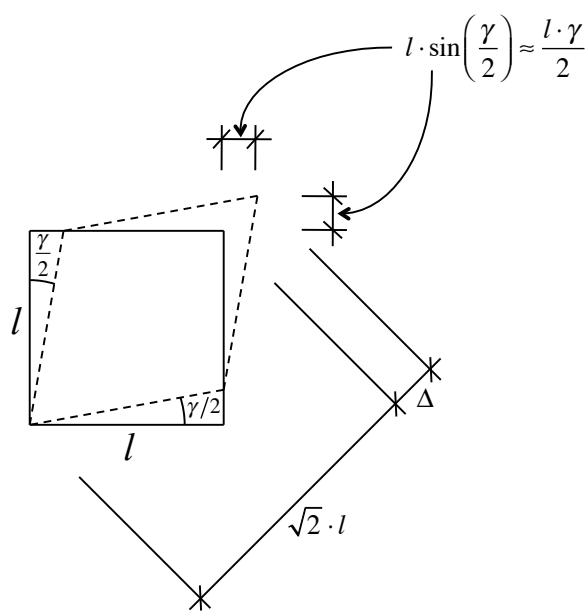

Figure 3: Derivation of the expression *G*.

The quantity Δ can be expressed in two ways. In the pure shear state:

$$\Delta = \sqrt{2 \cdot \left(\frac{l \cdot \gamma}{2} \right)^2} = \frac{l \cdot \gamma}{\sqrt{2}} \tag{18}$$

In the rotated state of pure axial stress:

$$\Delta = \varepsilon \cdot \left(\sqrt{2} \cdot l \right) = \left(\frac{\tau}{E} - \nu \cdot \frac{(-\tau)}{E} \right) \cdot \left(\sqrt{2} \cdot l \right) \tag{19}$$

Equating the two expressions for Δ yields:

$$\tau = \underbrace{\left(\frac{E}{2 \cdot (1+\nu)} \right)}_{\equiv G} \cdot \gamma \tag{20}$$

Hence, together with Eq. (16) the following equations complete the general Hooke's law:

$$\tau_{xy} = G \cdot \gamma_{xy} \quad , \qquad \tau_{yz} = G \cdot \gamma_{yz} \quad , \qquad \tau_{zx} = G \cdot \gamma_{zx} \tag{21}$$

In Voight notation it reads $\varepsilon_i = C_{ij}^{-1} \sigma_j$ or $\boldsymbol{\varepsilon} = \mathbf{C}^{-1} \boldsymbol{\sigma}$:

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} = \frac{1}{E} \cdot \left[\begin{array}{ccc|ccc} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 2(1+\nu) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(1+\nu) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(1+\nu) \end{array} \right] \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix} \tag{22}$$

Or, inversely $\sigma_i = C_{ij} \varepsilon_j$ or $\boldsymbol{\sigma} = \mathbf{C} \boldsymbol{\varepsilon}$:

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \cdot \left[\begin{array}{ccc|ccc} (1-\nu) & \nu & \nu & 0 & 0 & 0 \\ \nu & (1-\nu) & \nu & 0 & 0 & 0 \\ \nu & \nu & (1-\nu) & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} \end{array} \right] \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} \tag{23}$$

In index notation with the original strain and stress tensors, Hooke's law is written

$$\sigma_{ij} = \lambda \cdot \varepsilon_{kk} \cdot \delta_{ij} + 2 \cdot \mu \cdot \varepsilon_{ij} \tag{24}$$

where δ_{ij} is the unit matrix and λ and μ are the Lame parameters:

$$\mu = G \qquad \lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} \tag{25}$$

In addition to E, ν, G, μ, and λ, the bulk modulus, K, is employed in the study of volume change under hydrostatic pressure. Let $\varepsilon_{kk}=\varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33}$ denote the dilatation, i.e., the change in volume of an infinitesimally small cube. The pressure, p, is $\varepsilon_{kk}/3$. The bulk modulus relates the pressure to the dilatation: $p=-K\cdot\varepsilon_{kk}$, where

$$K = \frac{E}{3(1-2\nu)} \tag{26}$$

3D Elastic Beams

In other documents on this website, the Euler-Bernoulli and Timoshenko beam theories are described. Both those theories assume that "plane sections remain plane," which is not an exact description of reality. This document describes an alternative approach based on the general 3D theory of elasticity. These solutions tend to be more complex than simple beam theory, but they are often more accurate and provide further insight into the beam bending problem. In particular, by including the third dimension it is possible to study stresses in the cross-section. In fact, assumptions about the stresses represent the starting point for this theory. Letting x be the beam axis, as shown in Figure 4, it is assumed that the axial stresses σ_{yy} and σ_{zz}, as well as the shear stress τ_{yz}, are zero so that only the axial stress σ_{xx} and the shear stresses τ_{xy} and τ_{xz} appear in the beam (Timoshenko and Goodier 1969). In fact, it is assumed that the axial stress increases linearly with the distance from the centroid axis. In this regard, the notation from elementary beam theory is adopted to write the stress as

$$\sigma_{xx} = \frac{M}{I}\cdot z = \frac{P\cdot(L-x)}{I}\cdot z \tag{27}$$

where the notation I is employed in anticipation of the moment of inertia appearing in the theory.

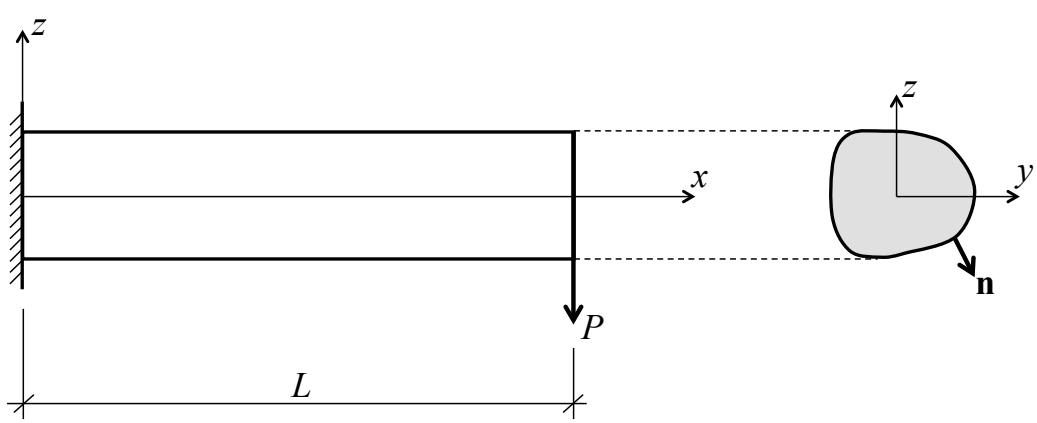

Figure 4: Beam considered in the 3D theory.

The general equations for stress equilibrium in the 3D theory of elasticity reads

$$\frac{d\sigma_{xx}}{dx}+\frac{d\tau_{yx}}{dy}+\frac{d\tau_{zx}}{dz}=0 \tag{28}$$

$$\frac{d\tau_{xy}}{dx}+\frac{d\sigma_{yy}}{dy}+\frac{d\tau_{zy}}{dz}=0 \tag{29}$$

$$\frac{d\tau_{xz}}{dx}+\frac{d\tau_{yz}}{dy}+\frac{d\sigma_{zz}}{dz}=0 \tag{30}$$

and with the previous stress-assumptions the equilibrium equations become

$$\frac{d\tau_{yx}}{dy}+\frac{d\tau_{zx}}{dz}=\frac{P\cdot z}{I} \tag{31}$$

$$\frac{d\tau_{xy}}{dx}=0 \tag{32}$$

$$\frac{d\tau_{xz}}{dx}=0 \tag{33}$$

In agreement with the solution technique for several other problems in elasticity, a stress function, $F(y,z)$, is now introduced. While that function does not have a physical meaning by itself, it is defined such that stresses are derived from it by differentiation. Specifically, the shear stress in the y-direction is obtained by differentiating F in the z-direction, and vice versa. By defining F such that

$$\tau_{xz}=\frac{dF}{dy}+\frac{P\cdot z^2}{2I}+h(y) \tag{34}$$

$$\tau_{xy}=-\frac{dF}{dz} \tag{35}$$

it becomes clear that the equilibrium equations in Eqs. (31), (32), and (33) are satisfied. Next, equations of kinematic compatibility, including material law, are introduced, as well as the following boundary conditions on the stress function. (This material is yet to be written here.) Because there are no shear stresses along the side-edges of the beam, the following equation must be satisfied:

$$\tau_{yx}\cdot\cos(\mathbf{n},y)+\tau_{zx}\cdot\cos(\mathbf{n},z)=0 \tag{36}$$

where $\cos(\mathbf{n},y)$ is the cosine of the angle between the y-axis and the normal vector to the cross-section edge, shown in Figure 4, and $\cos(\mathbf{n},z)$ is similarly defined. By defining a new axis, s, that follows the edge of the cross-section in the y-z-plane, as is common in the theories of shear and torsion of thin-walled beams, Eq. (36) becomes

$$\tau_{yx}\cdot\frac{dz}{ds}+\tau_{zx}\cdot\frac{dy}{ds}=0 \tag{37}$$

And here it stops for now; this section is unfortunately yet to be completed.

References

Timoshenko, S., and Goodier, J. N. (1969). *Theory of elasticity*. McGraw-Hill.