Stochastic Dynamics

This document addresses structures subjected to excitation from one or more continuous stochastic processes. As a result, the documents on structural dynamics and stochastic processes are important prerequisites for this material. In the document on dynamics the load is denoted $F(t)$ and the response by $u(t)$. That notation is translated into the processes $F(t)$ and $U(t)$ in this document. While the following derivations addresses only the structural dynamics problem, it is important to note that many other systems exist. Many of the concepts employed here, such as the transfer function to go from input to output, are valid also for systems beyond structural dynamics.

Time-domain Response

Another fact that is directly translated from the document on SDOF dynamics is the time-domain response of an SDOF system subjected to arbitrary excitation:

$$U(t) = \int_{-\infty}^{\infty} h(t-\tau) \cdot F(\tau) \cdot d\tau \tag{1}$$

where h is the unit impulse response function for the system. Several response quantities will be derived from this equation.

Mean Function

The mean function for the response is the expectation of Eq. (1)

$$\mu_U(t) = E\left[U(t)\right] = \int_{-\infty}^{t} h(t-\tau) \cdot E\left[F(\tau)\right] \cdot d\tau \tag{2}$$

If the mean of the excitation is constant at μ_F then the response mean is also constant:

$$\mu_U = \mu_F \cdot \int_{-\infty}^{\infty} h(t-\tau) \cdot d\tau = \mu_F \cdot H(0) \tag{3}$$

where $H(0)$ is the transfer function $H(\omega)$ evaluated at ω=0. This means that the response-mean is equal to the input-mean divided by the stiffness, K. In other words, the mean response is equal to the static response of the system subjected to the mean load. This also shows that if the input-mean is zero then the response-mean is zero as well.

Autocorrelation Function

The autocorrelation is obtained by utilizing its definition as an expectation, i.e., the mean square:

$$R_{UU}(\tau) = E\left[U(t) \cdot U(t+\tau)\right] = \cdots = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R_{FF}(\tau + r - s) \cdot h(r) \cdot h(s) dr\, ds \tag{4}$$

where r and s are auxiliary integration variables.

Frequency-domain Response

Response Spectrum

For a zero-mean process the autocorrelation function and the autocovariance function are identical, and the power spectral density (PSD) of the response is obtained by the Fourier transform of Eq. (4):

$$S_U(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R_{FF}(\tau)\cdot e^{-i\cdot\omega\cdot\tau}\,d\tau = \cdots = H(\omega)^*\cdot H(\omega)\cdot S_F(\omega) = \left|H(\omega)\right|^2\cdot S_F(\omega) \qquad (5)$$

where $H(\omega)^*$ is the complex conjugate of the complex transfer function and $|H(\omega)|$ is its modulus.

Cross-covariance and Cross-Spectrum

The cross-spectrum is the Fourier transform of the cross-covariance function:

$$S_{FU}(\omega) = \frac{1}{2\pi}\cdot\int_{-\infty}^{\infty} C_{FU}(\tau)\cdot e^{-i\omega\tau}\cdot d\tau \qquad (6)$$

For the complex cross-spectrum written on the standard form:

$$S_{FU}(\omega) = Co_{FU}(\omega) + i\cdot Qu_{FU}(\omega) \qquad (7)$$

the real part, *Co*, is called the co-spectrum and the imaginary part, *Qu*, is called the quadrature spectrum. For the complex cross-spectrum written on polar form:

$$S_{FU}(\omega) = \left|S_{FU}(\omega)\right|\cdot e^{-i\cdot\theta_{FU}(\omega)} \qquad (8)$$

the part $|S_{FU}(\omega)|$ is called the modulus spectrum and the part $\theta_{FU}(\omega)$ is called the phase spectrum. More can be said about these spectra, and this will be written later.

Overview

	Input		**Structure**		**Output**
Time-domain	$R_{FF}(\tau)$	→	$h(t)$ in Eq. (4)	→	$R_{UU}(\tau)$
	↑ Fourier transform ↓		↑ Fourier transform ↓		↑ Fourier transform ↓
Frequency-domain	$S_F(\omega)$	→	$H(\omega)$ in Eq. (5)	→	$S_U(\omega)$

Response to White Noise

White noise is characterized by a constant spectrum, $S_F(\omega)=S_o$. In other words, there is equal contribution from all possible frequencies. This type of excitation is entirely artificial, but it is more practically useful than it seems at first glance. This is because even narrowband excitation contributes merely around the natural frequency of vibration of the system; hence, the excitation spectrum can often be approximated as constant in

that vicinity, particularly for systems with low damping. Because the modulus of the complex transfer function is, from SDOF dynamics

$$|H(\omega)| = \frac{1}{K} \cdot \frac{1}{\sqrt{\left(1-\beta^2\right)^2 + \left(2 \cdot \xi \cdot \beta\right)^2}} \tag{9}$$

application of Eq. (5) leads to

$$S_U(\omega) = \frac{S_o}{K^2 \cdot \left(1 + (4\xi^2 - 2)\beta^2 + \beta^4\right)} \tag{10}$$

where β is, as in the dynamics documents, the ratio of the excitation frequency to the natural frequency of vibration for the system. The variance associated with this response process, i.e., the area underneath the spectrum is

$$\sigma_U^2 = \frac{\pi}{2} \cdot \frac{S_o}{M^2 \cdot \xi \cdot \omega_n^3} \tag{11}$$

Response to Narrowband Excitation

When damping in the system is low, the response is usually narrowband because most of the contributions come from the narrow range around the natural frequency of vibration. This means that the response spectrum is, according to Eq. (10):

$$S_U(\omega) = \frac{S_F(\omega_r)}{K^2 \cdot \left(1 + (4\xi^2 - 2)\beta^2 + \beta^4\right)} \tag{12}$$

where ω_r is the resonance frequency of the system. Mirroring deterministic dynamics, if the dominant excitation frequency hits the resonance frequency then the amplitude is "damping controlled." In contrast, if the excitation frequency is far lower than the resonance frequency, then the loading is essentially static and the response is "stiffness controlled." Conversely, if the excitation frequency is much higher than the excitation frequency then the load is not moving the mass much before changing direction, hence the stiffness is not activated and the response amplitude is "mass controlled."

Short-term Response Statistics

For design purposes it is of great interest to know "response statistics," such as the probability distribution of the maximum amplitude during a time-interval. Short-term statistics, sometimes called "local" analysis, looks at short time intervals and studies the probability distribution of "peaks." Long-term statistics, sometimes called "global" analysis and addressed in the next section, looks at longer time periods and studies the probability distribution of "extremes."

Up-crossing Rate

It is practically useful to know how often a stochastic process exceeds a selected threshold. In fact, the rate at which the process exceeds a threshold is a key building block for establishing the probability distribution of peaks and extremes. This rate is called "up-crossing rate" (out-crossing rate for vector processes) and for a zero-mean

process the down-crossing rate is essentially the up-crossing rate for the negative process. For the response process $U(t)$ the rate of up-crossing of the threshold r is defined as

$$\nu_U^+(r,t) = \lim_{\Delta t \to 0} \frac{P(\text{upcrossing of } r \text{ in } [t, t+\Delta t])}{\Delta t} \tag{13}$$

The probability in the numerator should formally include the possibility of more than one up-crossing, but for small Δt there is either zero or one crossings. The probability that an up-crossing takes place in the time-interval from t to $t+\Delta t$ can be obtained in several ways. One approach is to formulate an up-crossing event as the intersection of the following two events, illustrated in Figure 1a:

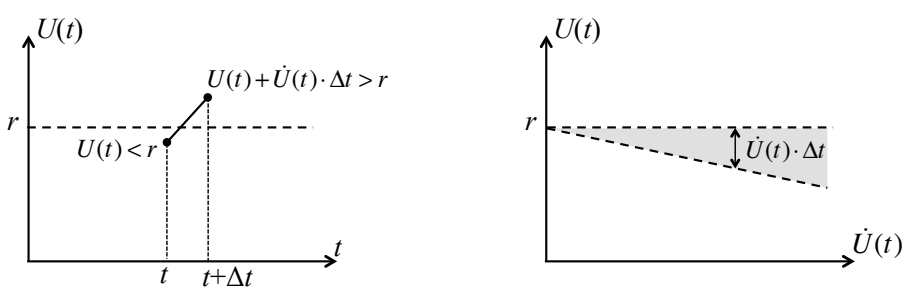
1. The response is below the threshold r at time t: $U(t) < r$
2. The response is above the threshold r at time $t+\Delta t$: $U(t) + \dot{U}(t) \cdot \Delta t > r$

where it is assumed that the velocity, $\dot{U}(t)$, is constant within Δt so that the amplitude changes by $\dot{U}(t) \cdot \Delta t$ within Δt. By moving that term over to the right-hand side of the inequality in the second event, the condition for an up-crossing can be written

$$r - \dot{U}(t) \cdot \Delta t < U(t) < r \tag{14}$$

This intersection event is visualized as a gray-shaded area in Figure 1b, where the abscissa axis is the velocity and the ordinate axis remains $U(t)$. As a result, the sought probability is obtained by integrating the joint PDF of $U(t)$ and $\dot{U}(t)$, here written $f_{U(t)\dot{U}(t)}(u,\dot{u})$, over the shaded area:

$$P(\text{upcrossing of } r \text{ in } [t, t+\Delta t]) = \int_0^\infty \int_{r-\dot{u}\cdot\Delta t}^{r} f_{U(t)\dot{U}(t)}(u,\dot{u})\, du\, d\dot{u} \tag{15}$$



(a) (b)

Figure 1: Visualization of the up-crossing event.

For infinitesimally small Δt, the amplitude $U(t)$ remains approximately equal to the threshold r throughout the shaded region in Figure 1b. As a result, the variable u can be replaced by the constant r in the joint PDF, making it constant in the $U(t)$-direction. Hence, Eq. (15) reads

$$P(\text{upcrossing of } r \text{ in } [t, t+\Delta t]) = \int_0^\infty (\dot{u} \cdot \Delta t) \cdot f_{U(t)\dot{U}(t)}(r,\dot{u})\, d\dot{u} \tag{16}$$

This is the general expression of the sought probability, which by substitution into Eq. (13) yields the following general expression for the up-crossing rate:

$$\nu_U^+(r,t) = \lim_{\Delta t \to 0} \frac{\int_0^\infty (\dot{u} \cdot \Delta t) \cdot f_{U(t)\dot{U}(t)}(r,\dot{u})\, d\dot{u}}{\Delta t} = \int_0^\infty \dot{u} \cdot f_{U(t)\dot{U}(t)}(r,\dot{u})\, d\dot{u} \tag{17}$$

As shown in the document on continuous stochastic processes, a stationary process $U(t)$ is uncorrelated with its derivative process $\dot{U}(t)$. Furthermore, for uncorrelated Gaussian random variables the joint PDF can be written as a product of the marginals, which means that lack of correlation implies independence. As a result, the up-crossing rate in Eq. (17) for a stationary Gaussian process is (Lutes and Sarkani 1997)

$$\begin{aligned}
\nu_U^+(r,t) &= \int_0^\infty \dot{u} \cdot f_{U(t)}(r) \cdot f_{\dot{U}(t)}(\dot{u})\, d\dot{u} \\
&= f_{U(t)}(r) \cdot \int_0^\infty \dot{u} \cdot f_{\dot{U}(t)}(\dot{u})\, d\dot{u} \\
&= \frac{1}{\sqrt{2\pi} \cdot \sigma_U} \cdot \exp\left(-\frac{1}{2} \cdot \left(\frac{r - \mu_U}{\sigma_U} \right)^2 \right) \cdot \int_0^\infty \dot{u} \cdot \frac{1}{\sqrt{2\pi} \cdot \sigma_{\dot{U}}} \cdot \exp\left(-\frac{1}{2} \cdot \left(\frac{\dot{u} - \mu_{\dot{U}}}{\sigma_{\dot{U}}} \right)^2 \right) d\dot{u} \\
&= \frac{1}{\sqrt{2\pi} \cdot \sigma_U} \cdot \exp\left(-\frac{1}{2} \cdot \left(\frac{r - \mu_U}{\sigma_U} \right)^2 \right) \cdot \left(\mu_{\dot{U}} \cdot \Phi\left(\frac{\mu_{\dot{U}}}{\sigma_{\dot{U}}} \right) + \frac{\sigma_{\dot{U}}}{\sqrt{2\pi}} \right)
\end{aligned} \tag{18}$$

where Φ is the standard normal CDF. For stationary response, the mean of the velocity is zero, so that

$$\nu_U^+(r,t) = \frac{\sigma_{\dot{U}}}{2\pi \cdot \sigma_U} \cdot \exp\left(-\frac{1}{2} \cdot \left(\frac{r - \mu_U}{\sigma_U} \right)^2 \right) \tag{19}$$

where it is noted that the rate of up-crossing of the mean is

$$\nu_U^+(r,t) = \frac{\sigma_{\dot{U}}}{2\pi \cdot \sigma_U} \tag{20}$$

Because the variance of a continuous stochastic process equals the spectral moment λ_0, and the variance of the derivative of a process equals the moment λ_2, Eq. (20) can be rewritten in terms of spectral moments as:

$$\nu_U^+(r,t) = \frac{1}{2\pi} \cdot \sqrt{\frac{\lambda_2}{\lambda_0}} \tag{21}$$

Rate of Occurrence of Peaks

A peak of $U(t)$ occurs when $\dot{U}(t) = 0$. As a result, the rate of occurrence of peaks equals the rate of down-crossings of the level zero by the derivative process. Similarly, the rate

of occurrence of valleys equals the rate of up-crossings of the level zero by the derivative process.

Probability Distribution of Peaks

The probability distribution of the random variable U_p is here sought, where U_p is the amplitude of an arbitrary peak. A simplified approach is to consider narrowband processes, where the number of up-crossings of a level equals the number of peaks above that level. In that case, the total number of peaks equals the number of up-crossings of the mean level, and the number of peaks above the threshold u_p equals the number of up-crossings of the level u_p. Using up-crossing rates, the probability of peaks above the threshold u_p is then

$$P(U_p > u_p) = \frac{\nu^+(u_p)}{\nu^+(0)} \tag{22}$$

hence, the sought CDF is

$$F(x_p) = 1 - \frac{\nu^+(x_p)}{\nu^+(0)} \tag{23}$$

Substitution of Eq. (19) for Gaussian processes yields, for a zero-mean process:

$$\begin{aligned} F(u_p) &= 1 - \frac{\dfrac{\sigma_{\dot{U}}}{2\pi \cdot \sigma_U} \cdot \exp\left(-\dfrac{1}{2} \cdot \left(\dfrac{r - \mu_U}{\sigma_U} \right)^2 \right)}{\dfrac{\sigma_{\dot{U}}}{2\pi \cdot \sigma_U} \cdot \exp\left(-\dfrac{1}{2} \cdot \left(\dfrac{0}{\sigma_U} \right)^2 \right)} \\ &= 1 - \exp\left(-\frac{1}{2} \cdot \left(\frac{u_p}{\sigma_U} \right)^2 \right) \end{aligned} \tag{24}$$

Differentiation yields the corresponding PDF:

$$f(u_p) = \frac{dF(u_p)}{du_p} = \frac{u_p}{\sigma_U^2} \cdot \exp\left(-\frac{u_p^2}{2 \cdot \sigma_U^2} \right) \tag{25}$$

which is the Rayleigh distribution. The mean of this distribution, i.e., the expected peak response is, from the Rayleigh distribution in the document on continuous random variables:

$$\mu_{u_p} = \sigma_U \cdot \sqrt{\frac{\pi}{2}} \tag{26}$$

From the same document the standard deviation is

$$\sigma_{u_p} = \sigma_U \cdot \sqrt{\frac{4 - \pi}{2}} \tag{27}$$

Long-term Response Statistics

Consider a random variable defined as

$$U_e(T) = \max_{0 \le t \le T}\left(U(t)\right) \tag{28}$$

namely the extreme value of a stochastic process during the time interval T. In practical applications, the probability distribution of U_e is perhaps the most important response statistic. Establishing this distribution in general is a difficult problem that is related to the challenging first-passage problem in time-variant reliability analysis. A simple and sometimes accurate approach for narrowband processes is to assume that up-crossings of a high threshold are independent of each other. In that case, the Poisson process can be employed to model the up-crossing events. The number of up-crossings during T is then given by the Poisson distribution with rate of up-crossings $\nu_U^+(u_e)$ over the threshold u_e:

$$p(n) = \frac{(\nu_U^+(u_e)\cdot T)^n}{n!} e^{-\nu_U^+(u_e)\cdot T} \tag{29}$$

Because the probability of zero up-crossings is $p(0)$, the probability of any number of up-crossings is $1-p(0)$. As a result, the CDF (the probability of extremes less than the threshold) is

$$F(u_e) = p(0) = \exp\left(-\nu_U^+(u_e)\cdot T\right) \tag{30}$$

Substituting the crossing rate for stationary Gaussian processes yields:

$$F(u_e) = p(0) = \exp\left(-\frac{\sigma_{\dot{U}}}{2\pi\cdot\sigma_U}\cdot\exp\left(-\frac{1}{2}\cdot\left(\frac{u_e-\mu_U}{\sigma_U}\right)^2\right)\cdot T\right) \tag{31}$$

From this distribution it is possible to derive the mean and variance of the extreme response during time T:

$$\mu_{u_e} = \int_0^\infty u_e\cdot\frac{dF(u_e)}{du_e}\cdot du_e \approx \sigma_U\cdot\left(\sqrt{2\cdot\ln\left(\frac{1}{2\pi}\cdot\frac{\sigma_{\dot{U}}}{\sigma_U}\cdot T\right)} + \frac{\gamma}{\sqrt{2\cdot\ln\left(\frac{1}{2\pi}\cdot\frac{\sigma_{\dot{U}}}{\sigma_U}\cdot T\right)}}\right) \tag{32}$$

where Euler's constant is γ=0.5772. Furthermore, the second moment is

$$E\left[u_e^2\right] = \int_0^\infty u_e^2\cdot\frac{dF(u_e)}{du_e}\cdot du_e \approx 2\cdot\sigma_U^2\cdot\left(\ln\left(\frac{1}{2\pi}\cdot\frac{\sigma_{\dot{U}}}{\sigma_U}\cdot T\right)+\gamma\right) \tag{33}$$

which means that the variance is

$$\sigma_{u_e}^2 = E[u_e^2] - \mu_{u_e}^2 \approx \frac{\pi^2\cdot\sigma_U^2}{12\cdot\ln\left(\frac{1}{2\pi}\cdot\frac{\sigma_{\dot{U}}}{\sigma_U}\cdot T\right)} \tag{34}$$

It is emphasized that the expected extreme value, μ_{u_e}, has a relatively high chance, say around 50%, of being exceeded. It is also noted that the expected extreme value increases with *T*, while the variance diminishes. The probability distribution above can also be used to obtain an expression for the threshold *r* that has probability *p* of *not* being exceeded during the time interval *T*:

$$r(p,T) = \sigma_U \cdot \sqrt{2 \cdot \ln\left(\frac{\frac{1}{2\pi} \cdot \frac{\sigma_{\dot{U}}}{\sigma_U} \cdot T}{\ln\left(\frac{1}{p}\right)} \right)} \tag{35}$$

Response of MDOF Systems

This is yet to be written, where the transfer functions are collected in a matrix and the unit impulse response functions are collected in a vector:

$$\mathbf{H}(\omega) = \left[H_{ij}(\omega) \right] \tag{36}$$

$$\mathbf{h}(t) = \left[h_{ij}(t) \right] \tag{37}$$

Nonlinear Structures and Non-Gaussian Loading

The objective in general time-variant reliability analysis is to determine the probability of limit-state violation during the time period *T*:

$$p_f(T) = P\left(\left\{ \min_{0 \le t \le T} g(\mathbf{x}(t),t) \right\} \le 0 \right) \tag{38}$$

A brute-force approach to solving this problem is to define a limit-state function at "every" time instant and treat the problem as a series system reliability problem. Another approach to solving the first-passage problem is to first consider the event that the limit-state function is positive at time *t*:

$$g(\mathbf{x}(t),t) > 0 \tag{39}$$

Next, consider the event that the limit-state function is negative at time $t+\Delta t$:

$$g(\mathbf{x}(t+\Delta t),t+\Delta t) \le 0 \tag{40}$$

One or more out-crossings have occurred during Δt. Related to Eqs. (39) and (40), consider two auxiliary limit-state functions. The first contains the limit-state function at time *t*:

$$g_1 = -g(\mathbf{x}(t),t) \tag{41}$$

The second is the limit-state function at $t+\Delta t$ represented by its linear Taylor approximation centred at *t*:

$$g_2 = g(\mathbf{x}(t),t) + \frac{\partial g(\mathbf{x}(t),t)}{\partial t} \cdot \Delta t \tag{42}$$

In terms of the auxiliary limit-state functions, the probability of out-crossing is now formulated as a parallel system problem

$$P\left(g_1 \le 0 \bigcap g_2 \le 0\right) \tag{43}$$

so that the mean crossing rate is (Hagen and Tvedt 1991)

$$\nu(t) = \lim_{\Delta t \to 0} \frac{P\left(g_1 \le 0 \bigcap g_2 \le 0\right)}{\Delta t} \tag{44}$$

An exact solution, in terms of an integral, is available for the bi-variate normal distribution that is required for the two-component parallel system probability. However, care must be exercised because of the high correlation between g_1 and g_2. The upper bound to the probability of excursion into the failure domain during time T is

$$p_f(T) = \int_0^T \nu(t)\,dt \tag{45}$$

In situations where the crossing events are independent, the Poisson process can again be employed and the probability of one or more crossings is

$$p_f(T) = 1 - e^{\int_0^T \nu(t)\,dt} \tag{46}$$

References

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Lutes, L. D., and Sarkani, S. (1997). *Stochastic analysis of structural and mechanical vibrations*. Prentice Hall.