# Set Theory

## Definitions

Set theory is a language with which we can talk about events. In this sense, it underpins the theory of probability, where the likelihood of events is in question.

- The set of all possible events is called the sample space, $S$
- Each individual sample point is denoted $x$
- Any collection of sample points is called an event, typically denoted $E$, perhaps with a subscript to further identify the event
- The complement of an event, denoted $\bar{E}$, is an event that contains all the sample points that are not in $E$.
- The event S is called the certain event because it is bound to occur
- The event ø is the null event; it contains no sample points and has no possibility of occurring.

There are two types of sample spaces:

- Discrete (finite or infinite)
- Continuous

## Venn Diagrams

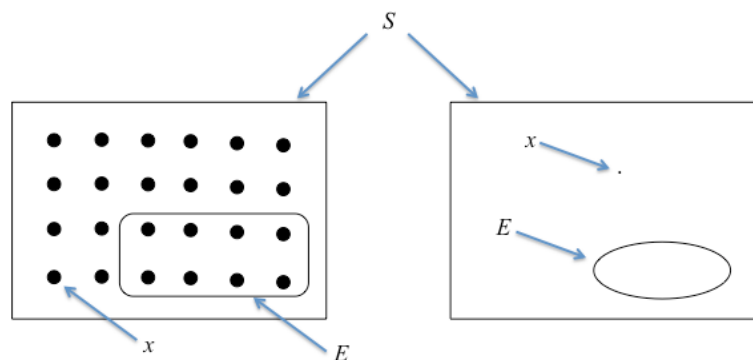Venn diagrams are useful for visualizing events, as shown in Figure 1.



**Figure 1: Venn diagram for discrete (left) and continuous (right) sample spaces.**

## Operations

The union and intersection operations create compound events out of two or more events. The union of two events, denoted $E_1 \bigcup E_2$, is an event that contains all the sample points that are either in $E_1$ or in $E_2$. It is therefore pronounced "or." The intersection of two events, denoted $E_1 \bigcap E_2$ or $E_1 E_2$ in shorthand notation is an event that contains all the sample points that are both in $E_1$ and in $E_2$. It is therefore pronounced "and." For a

continuous sample space the union and intersection operations are grey-shaded in the Venn diagrams in Figure 2.
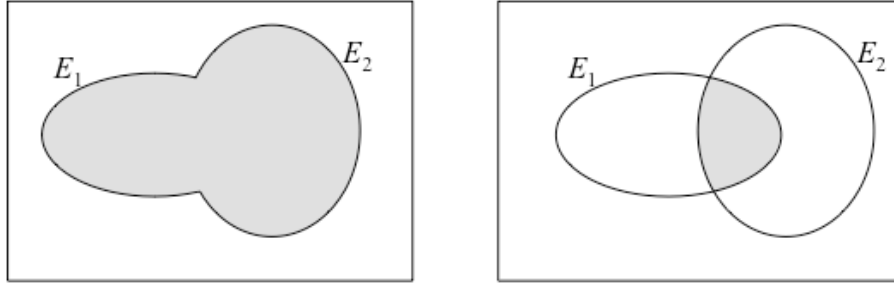


**Figure 2: Visualization of union and intersection events.**

The union and intersection operators obey the following rules:

- Commutative rule:

$$E_1 \cup E_2 = E_2 \cup E_1 \tag{1}$$

$$E_1 \cap E_2 = E_2 \cap E_1 \tag{2}$$

- Associative rule:

$$\left(E_1 \cup E_2\right) \cup E_3 = E_1 \cup \left(E_2 \cup E_3\right) \tag{3}$$

$$\left(E_1 \cap E_2\right) \cap E_3 = E_1 \cap \left(E_2 \cap E_3\right) \tag{4}$$

- Distributive rule:

$$\left(E_1 \cup E_2\right) \cap E_3 = \left(E_1 \cap E_3\right) \cup \left(E_2 \cap E_3\right) \tag{5}$$

$$\left(E_1 \cap E_2\right) \cup E_3 = \left(E_1 \cup E_3\right) \cap \left(E_2 \cup E_3\right) \tag{6}$$

To avoid excessive use of parentheses, in the same way as multiplication takes precedence over addition, intersection takes precedence over union operations.


# MECE

The terms "mutually exclusive" and "collectively exhaustive" are often used:

- Two events are said to be mutually exclusive if their intersection is the null event
- Two events are collectively exhaustive if their union constitutes the entire sample space


# De Morgan's Rules

The following rules allow us to transform union of events into intersection of events, and vice versa. The fundamental version of de Morgan's rule states: "the complement of a union is equal to the intersection of the complements:"

$$\overline{E_1 \bigcup E_2} = \bar{E}_1 \bigcap \bar{E}_2 \tag{7}$$

This rule is verified by visualizing both sides of Eq. (7) in a Venn diagram, where it is observed that they both refer to the same event. The rule is generalized to

$$\overline{E_1 \bigcup E_2 \bigcup \cdots \bigcup E_n} = \bar{E}_1 \bar{E}_2 \cdots \bar{E}_n \tag{8}$$

by repeated application of Eq. (7). Another version of de Morgan's rule, i.e., another rule, states: "the complement of an intersection is equal to the union of the complements:

$$\overline{E_1 \bigcap E_2} = \bar{E}_1 \bigcup \bar{E}_2 \tag{9}$$

This rule is derived by applying Eq. (7) to two complements:

$$\overline{\bar{E}_1 \bigcup \bar{E}_2} = \bar{\bar{E}}_1 \bigcap \bar{\bar{E}}_2 = E_1 \bigcap E_2 \tag{10}$$

Taking the complement on both sides yield Eq. (9).