Moment Distribution Method

The moment distribution method—sometimes named the Cross method after its inventor Hardy Cross—plays a special role in structural engineering. It is a hand calculation method for the analysis of statically indeterminate frame structures. Some say the method is obsolete after the advent of computer methods. Others say it is essential and powerful as an approach that prevents black-box use of structural analysis programs. In this document the derivation, capability, and limitations of the method are established.

Formally, the moment distribution method is a displacement method, because it is founded on equilibrium equations rather than compatibility equations. Practically, it is based on iterative clamping and releasing of joints. To understand the method, consider the end moments of frame members coming into a joint. Equilibrium requires that the sum of these moments is zero. The moment distribution method iteratively applies moment equilibrium at joints. Consider a structure in which only joint rotations, not joint displacements, are unknown. Such structures are said to have zero "sidesway." The archetypical case addressed by moment distribution is a continuous horizontal beam with any number of supports.

To understand the method, first imagine temporary clamps at all joints. The clamps prevent all joints from rotating. Next, remove the clamps one-by-one, i.e., joint-by-joint. At each joint, the member-end moments generate an unbalanced moment that is distributed to each member-end according to the relative bending stiffness of each member. This is the essence of the moment distribution method. This method is ideally suited for horizontal continuous beam members because they allow us to keep track of the moment equilibrium iterations in a neat table below the structure.

To derive the method, consider two beam members *AB* and *BC* that are continuously attached at *B*. This configuration is shown in Figure 1. Each beam may support some loads that, in the situation of clamped member-ends, give rise to the fixed-end moments FEM_{AB}, FEM_{BA}, FEM_{BC}, FEM_{CB}, where the two indices indicate the member and the first index indicates the location of the moment. The convention is that clockwise end moments are positive. Fixed-end moments for common loading scenarios are provided in another document. Now define the "unbalanced moment" at joint *B* as

$$UM_B = FEM_{BA} + FEM_{BC} \tag{1}$$

Both moments act positively clockwise on the member-ends and, thus, counter clockwise on the joint. In other words, if both fixed-end moments were positive then they would rotate the joint in the counter-clockwise direction when the joint is unclamped. In terms of moments, when the clamp is released they distribute onto the member ends according to the relative bending stiffness of each member-end. This is expressed in terms of a "distribution factor," called *DF*. As shown in Figure 1, the distributed end moments are

$$M_{BA} = -\left(DF_{BA} \cdot UM_B\right) \tag{2}$$

$$M_{BC} = -\left(DF_{BC} \cdot UM_B\right) \tag{3}$$

where DF_{BA} and DF_{BC} are distribution factors that are proportional to the bending stiffness of each member. In other words, the distribution factor DF_{BA} identifies how much of the unbalanced end moment will be attracted to end B of the member that connects A and B. The sum of the distribution factors in a joint equals unity.

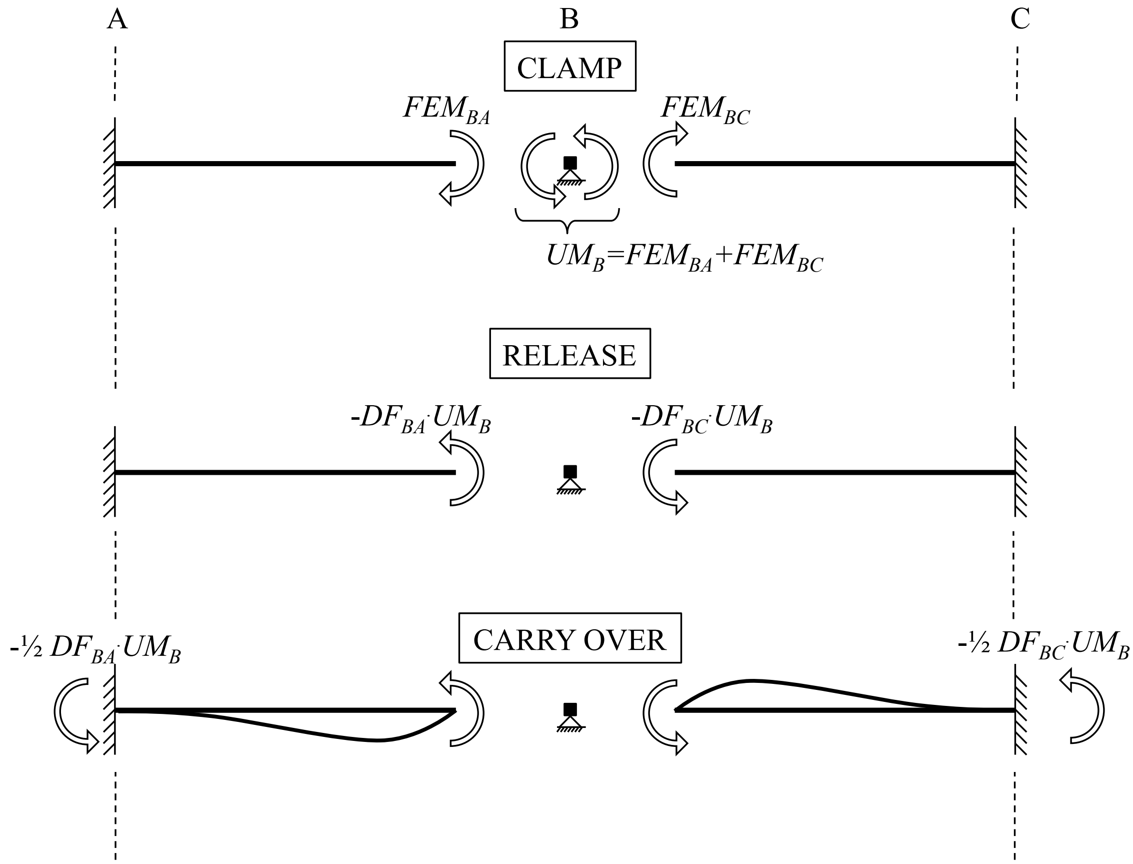

Figure 1: Key concepts of the moment distribution method.

To determine the value of the distribution factors, first express the rotation of each member-end as a function of the unbalanced moment. Utilizing the relationship between moment and rotation at the end of a fixed-fixed member, provided for example by the slope-deflection equation, one obtains:

$$\theta_{BA} = \frac{-\left(DF_{BA} \cdot UM_B\right) \cdot L_{AB}}{4EI_{AB}} \tag{4}$$

$$\theta_{BC} = \frac{-\left(DF_{BC} \cdot UM_B\right) \cdot L_{BC}}{4EI_{BC}} \tag{5}$$

Continuity of the beam, which requires that $\theta_{BA}=\theta_{BC}$, in combination with the condition that $DF_{BA}+DF_{BC}=1$ yields:

$$DF_{BA} = \frac{4EI_{AB}/L_{AB}}{4EI_{AB}/L_{AB} + 4EI_{BC}/L_{BC}} \tag{6}$$

$$DF_{BC} = \frac{4EI_{BC}/L_{BC}}{4EI_{AB}/L_{AB} + 4EI_{BC}/L_{BC}} \tag{7}$$

From these equations we understand that the generic equation to determine the distribution factors is

$$DF_i = \frac{4EI_i / L_i}{\sum 4EI / L} \tag{8}$$

where the sum in the denominator represents the sum of the bending stiffness of all members coming into a joint. Having distributed the unbalanced moment in an unclamped joint according to the distribution factors in Eq. (8), the next step is to recognize that the distributed moments at *B* travel to the opposite member-end according to the equations

$$COM_{BA} = \frac{1}{2} \cdot DF_{BA} \cdot UM_B \tag{9}$$

$$COM_{BC} = \frac{1}{2} \cdot DF_{BC} \cdot UM_B \tag{10}$$

where *COM* is shorthand notation for “carry-over moment.” The factor ½ is obtained from the slope-deflection equation by considering a beam that is rotated at one end and held fixed at the other end. In short, an applied end moment translates to half the value at the opposite fixed end.

That concludes the theoretical foundation for the moment distribution method. As conceptually shown in Figure 2, moment distribution is carried out as follows:

1. First, draw a sketch of the structure
2. Under each joint, make room for one column of numbers per member-end
3. The first row of each column in the table contains the distribution factor, *DF*; it is computed according to Eq. (8)
4. The second row contains the fixed-end moment, *FEM*; it is computed by any structural analysis method and a look-up table is provided in an auxiliary document in these notes on structural analysis
5. The subsequent rows contain either distributed end moments, *DEM*, or carry-over moments, *COM*, according to the following steps
6. Select any joint and compute the unbalanced moment. It is the sum of all fixed-end moments and externally applied moments in the joint. Clockwise fixed-end moments are positive. Un-clamp the joint and distribute the *negative* unbalanced moment to each member-end according to its distribution factor. The negative sign is applied because a positive unbalanced moment, which drives the joint counter clockwise, applies a negative moment to the member-ends when the joint is unclamped.
7. Send carry-over moments, i.e., half the distributed end moment, to the adjacent joints and re-clamp the joint under consideration
8. Move to the next joint and repeat Steps 6 and 7

9. The analysis is terminated when there are no more unbalanced moments to release, or when the unbalanced moments reach values that are below the desired level of accuracy. At that time, add a final row with the sum of all the moments (*FEM*, *DEM*, *COM*) of that column.

To draw the final bending moment diagram, start by drawing the ordinates provided by the end moments in the final row of the moment distribution table. Then add the shape of the bending moment diagram between the member-ends, i.e., the moment diagram for the member as if it was simply supported. It is possible to carry out the moment distribution method also for structures with side-sway. However, in this circumstance the method looses its appeal, which means that computational methods, such as the stiffness method, are preferred.

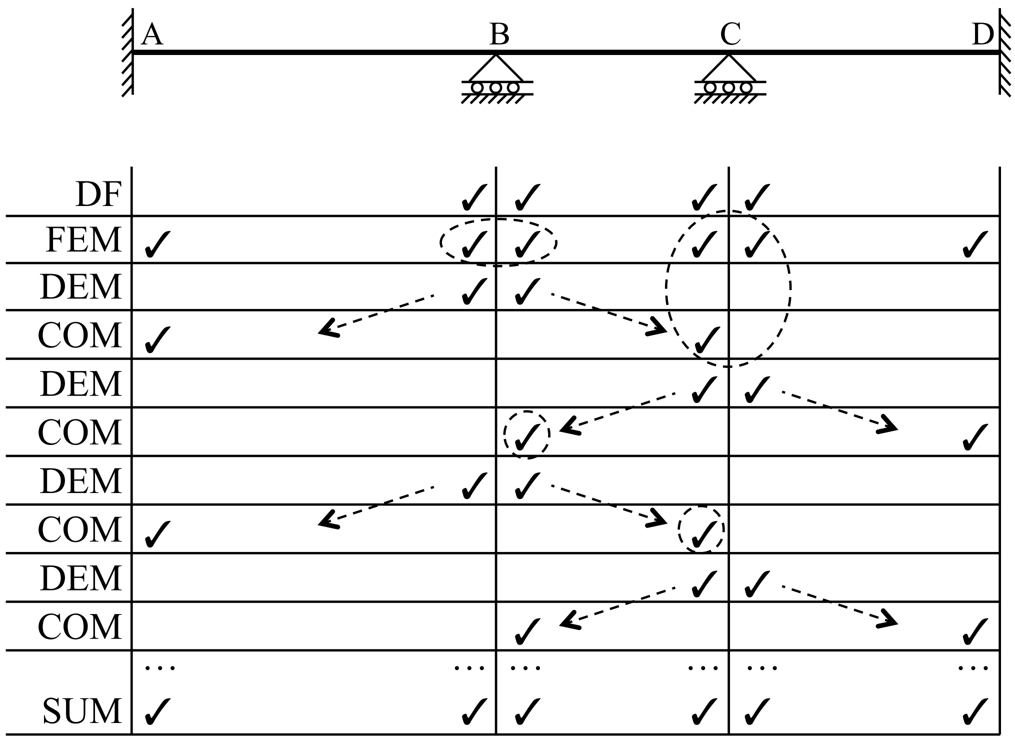

Figure 2: Moment distribution iterations.

Modified Procedures

It is possible to introduce modifications of the distribution factors, *DF*, and the fixed-end moments, *FEM*, which makes the moment distribution converge quicker. In particular, the distribution factors can be modified for symmetric structures and for members that have one pin or roller end. Similarly, fixed-end moments can be developed for members that have only one end fixed, while the other is pinned. In this document it is preferred to avoid modifications of fixed-end moments in order to maintain only one table of fixed-end moments. On the other hand, it is recognized that the modification of distribution factors can save significant time in hand calculations. Therefore, modified distribution factors, i.e., modification of Eq. (8) are established for symmetric members and for members that have one pin or roller end.

End with Pin or Roller

To derive a modified version of Eq. (8) when member BC has a pin or roller at C, consider the beam in the middle rectangle in Figure 3. In this situation the unit virtual load method reveals the end moment at B is equal to $M_{BC}=3EI/L$. As a result, the distribution factors in Eqs. (6) and (7) when member BC has a pin or roller at C are

$$DF_{BA} = \frac{4EI_{AB}/L_{AB}}{4EI_{AB}/L_{AB} + 3EI_{BC}/L_{BC}} \tag{11}$$

$$DF_{BC} = \frac{3EI_{BC}/L_{BC}}{4EI_{AB}/L_{AB} + 3EI_{BC}/L_{BC}} \tag{12}$$

Symmetry

To derive a modified version of Eq. (8) for a symmetric member, i.e., one that crosses a symmetry line, consider the bottom rectangle in Figure 3. Due to symmetry the rotation at C is the same as the rotation at B, albeit counter-clockwise. In this situation, the slope-deflection equation yields the end moment at B equal to $M_{BC}=2EI/L$, as shown at the bottom of Figure 3. As a result, the distribution factors in Eqs. (6) and (7) when member BC is symmetric are

$$DF_{BA} = \frac{4EI_{AB}/L_{AB}}{4EI_{AB}/L_{AB} + 2EI_{BC}/L_{BC}} \tag{13}$$

$$DF_{BC} = \frac{2EI_{BC}/L_{BC}}{4EI_{AB}/L_{AB} + 2EI_{BC}/L_{BC}} \tag{14}$$

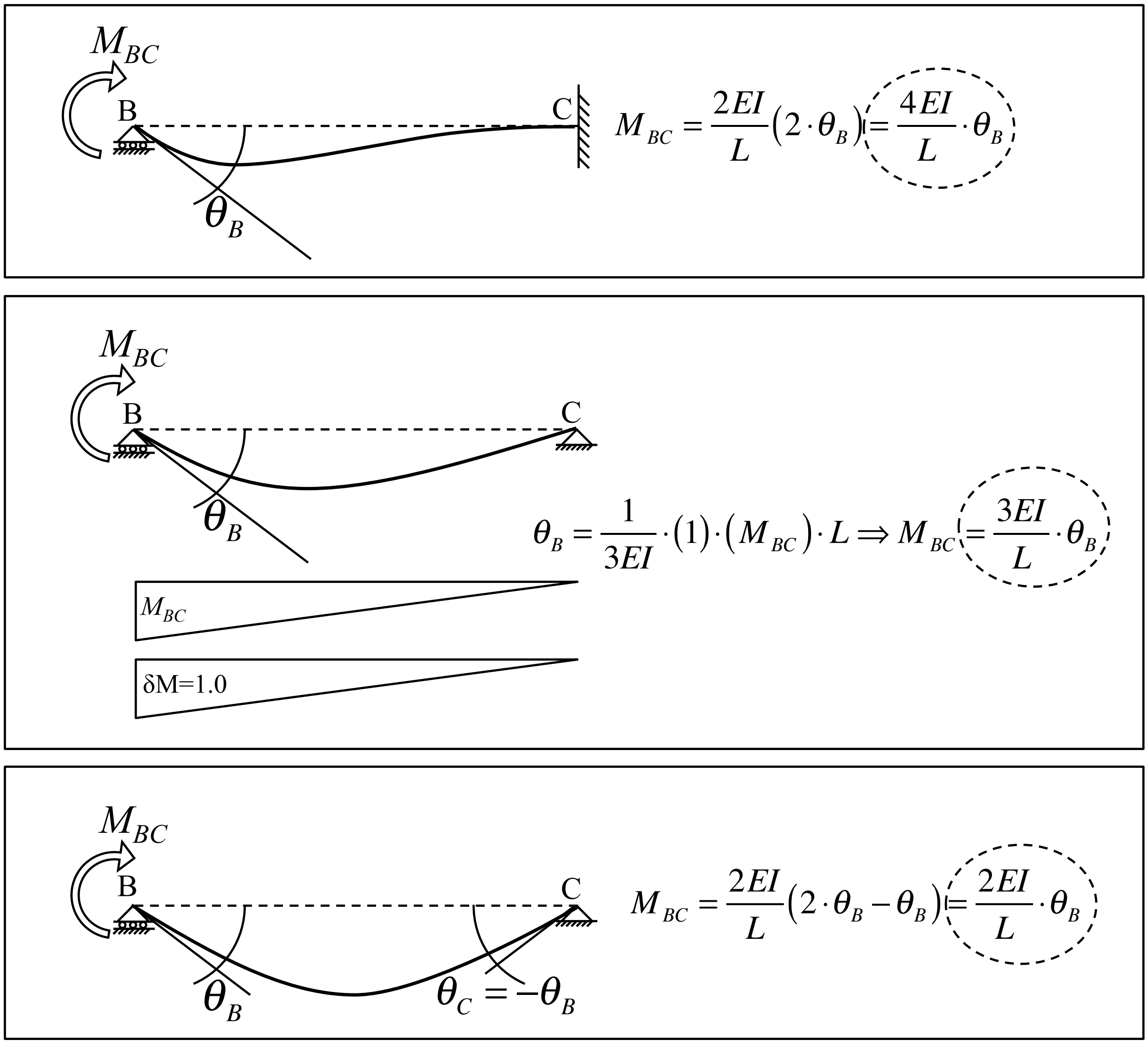

Figure 3: Derivation modified distribution factors.