# Flexibility Method

The flexibility method is categorized as a force method because it regards the forces in the structure as the primary unknowns. In contrast to displacement methods, in which equilibrium equations are postulated to determine the unknown deformations, the core of the flexibility method is the formulation of compatibility equations. The key steps of the method are:

1. Determine the structure's degree of static indeterminacy
2. Make the structure statically determinate by introducing releases
3. Establish compatibility equations to ensure continuity at the releases
4. Solve the compatibility equations and draw the final section force diagram (M, V, N) for each member

Usually, the number of releases equals the degree of static indeterminacy. Exceptions are when some forces are known in particular load cases. For trusses, it is common to remove supports and cut members to introduce releases. For frame members, it is common to remove supports, release bending moments by introducing hinges, and sometimes to release shear and axial forces.

The displacements or rotations that take place along the releases are called "gaps." The gap that opens due to the actual loads on the structure is called $\Delta_{i0}$, where the index $i$ indicates the location of the release, while 0 says it is due to the actual loads. Furthermore, let $\Delta_{ij}$ denote the gap that opens at location $i$ due to a unit load at location $j$. These are called flexibility coefficients because they reveal the force it takes to increase/decrease the gap. Compatibility equations serve the purpose of closing the gaps. In index notation, they read

$$\Delta_{i0} + \Delta_{ij} \cdot x_j = 0 \tag{1}$$

where $x_j$ are the unknown forces (axial force, bending moment, or shear force) at the releases. In matrix notation, the compatibility equations read

$$\mathbf{d} + \mathbf{f}\mathbf{x} = \mathbf{0} \tag{2}$$

where $\mathbf{d}$ is the vector of gaps due to the actual forces, $\mathbf{f}$ is the matrix of flexibility coefficients, and $\mathbf{x}$ is the vector of unknown forces at the releases. For a structure with one degree of static indeterminacy, where a force at location "A" is selected as the redundant, the compatibility equation reads

$$\Delta_{A0} + \Delta_{AA} \cdot x_A = 0 \tag{3}$$

For a structure with two degrees of static indeterminacy, where a force at location "A" and a force at location "B" are selected as redundants, the compatibility equations read:

$$\begin{aligned} \Delta_{A0} + \Delta_{AA} \cdot x_A + \cdot \Delta_{AB} \cdot x_B = 0 \\ \Delta_{B0} + \Delta_{BA} \cdot x_A + \Delta_{BB} \cdot x_B = 0 \end{aligned} \tag{4}$$

In summary, there are two key steps to establish the compatibility equations: First select releases that make the structure statically determinate and then determine the deformations $\Delta_{i0}$ and $\Delta_{ij}$. The first step is addressed on an ad hoc basis. This is the reason why the flexibility method is rarely implemented on the computer; it is difficult to set up a generic algorithm to select releases for any statically indeterminate structure. The second step is usually solved by the unit virtual load method. I.e., the deformations $\Delta_{i0}$ and $\Delta_{ij}$ are determined by applying unit virtual forces along the releases. In this document, the section force diagrams due to a unit force along release number $j$ are denoted $M_j$, $V_j$, and $N_j$. Upon solving the compatibility equations in Eq. (1) for $x_j$ the final section force diagrams are determined by combining the diagrams for the statically determinate auxiliary structure:

$$\begin{aligned} M &= M_0 + M_A \cdot x_A + M_B \cdot x_B \\ V &= V_0 + V_A \cdot x_A + V_B \cdot x_B \\ N &= N_0 + N_A \cdot x_A + N_B \cdot x_B \end{aligned} \tag{5}$$

where $M_0$ is the bending moment diagram for the statically determinate structure subjected to the applied loads, $\delta M_A$ is the bending moment diagram for the statically determinate structure due to a unit virtual load at $A$, and so forth.

## Settlements and Changes in Member Lengths

With one exception, the effect of settlements and changes in member lengths are included in the left-most terms in the left-hand side of the compatibility equations. Specifically, the additional "gap openings" due to settlements and length changes are added to $\Delta_{i0}$. These are conveniently computed by the unit virtual load method, as described towards the end of the document on that method. The one exception appears when the settlement takes place exactly at the redundant. For example, if a support reaction is selected as the redundant and that support settles, then this settlement is placed at the right-hand side of the compatibility equation instead of zero. If the settlement is in the same direction as the positive direction of the redundant then the settlement value is entered with a positive sign in the right-hand side.

An interesting case of member length change is post-tensioning. As an example, think of a truss member that is subjected to post-tensioning by some bar tensioning mechanism, after the truss is built. Suppose the forces and deformations in the structure are sought under these circumstances. The flexibility method addresses this problem as follows:

1. Consider the structure without any external loads
2. Make the structure statically determinate by introducing cuts, not necessarily in the post-tensioned members
3. Compute the gaps $\Delta_{i0}$ due to a unit shortening of the post-tensioned member by the unit virtual load method
5. As usual, determine the flexibility coefficients, establish the compatibility equations, solve them, and draw the final section force diagram ($M$, $V$, $N$) for each member; this provides the relationship between the unit member shortening and the member forces
6. If the deformations are sought, re-analyze the structure for a unit load at the location where the displacement or rotation is sought and apply the unit virtual

load method; this provides the relationship between the unit member shortening and the sought deformation