# Degree of Static Indeterminacy

The degree of static indeterminacy (DSI) of a structure tells us which methods are suitable for analysing it. If a structure is statically determinate, i.e., DSI=0, then it suffices to use equilibrium to determine the section forces. In fact, this is the fundamental definition of DSI: Any nonzero degree of indeterminacy means that there are too many unknown forces for us to determine the section forces by equilibrium. In such situations, when DSI>0, then more complex force-based or displacement-based methods must be used. The primary objective in this document is to describe techniques for computing the DSI for truss and frame structures. Two additional concepts are briefly described: stability and degrees of freedom.

## DSI by Counting Members and Joints

Regardless of which approach is used to determine the DSI, the fundamental objective is to count unknown forces and compare it to the number of equilibrium equations. Any surplus of unknown forces equals the DSI. The forces include axial forces, shear forces, and bending moments in the members, as well as reaction forces at supports. Any hinge or other forms of release of member forces reduce the number of unknowns. Conversely, the equilibrium equations are established at the member ends, i.e., at the joints. The following formula captures those statements into one expression, valid for all kinds of 2D and 3D truss and frame structures:

$$DSI = \left( f \cdot m + s \right) - \left( e \cdot j + h \right) \tag{1}$$

where all variables are non-negative integers with the following meaning:

| | | | |
|---|---|---|---|
| $f$ | = | forces | = number of internal force in each member |
| $m$ | = | members | = number of members |
| $s$ | = | support reactions | = number of support reactions, often several per support |
| $e$ | = | equations | = number of equilibrium equations per joint |
| $j$ | = | joints | = number of joints |
| $h$ | = | hinges | = number of moment hinges or other section force releases |

The number of internal forces, $f$, in each member depends on the member type. A truss member has only one unknown force: the axial force. Conversely, a frame member in a 2D structural model has three internal forces: axial force, shear force, and bending moment. This number increases from three to six for 3D frame members. Table 1 summarizes the value of $f$ for different structures. The number of equilibrium equations, $e$, per joint is also dependent on the type of structure, and is obtained by counting the directions in which equilibrium can be considered. For 2D frame structures there are three equilibrium equations per joint: horizontal, vertical, and angular equilibrium. For a joint in a 2D structure with only truss members entering, i.e., member without bending stiffness, rotational equilibrium is cancelled. For mixed joints, where at least one frame member enters, three equilibrium equations are available. Table 1 summarizes the value of $e$ for different structures.

**Table 1: Forces per member and equations per joint.**

| | *f* | *e* |
|---|---|---|
| 2D truss | 1 | 2 |
| 2D frame | 3 | 3 |
| 3D truss | 1 | 3 |
| 3D frame | 6 | 6 |

The number of support reactions, *s*, is shown Figure 1 for different support types. The arrows in the figure show the forces when the supports are applied to 2D structures. The corresponding number of support reactions in 3D structures is somewhat subjective, in the sense that the number depends on how the engineer designs the support. Thus, great caution must be exercised when applying the numbers in Figure 1 to 3D structures.

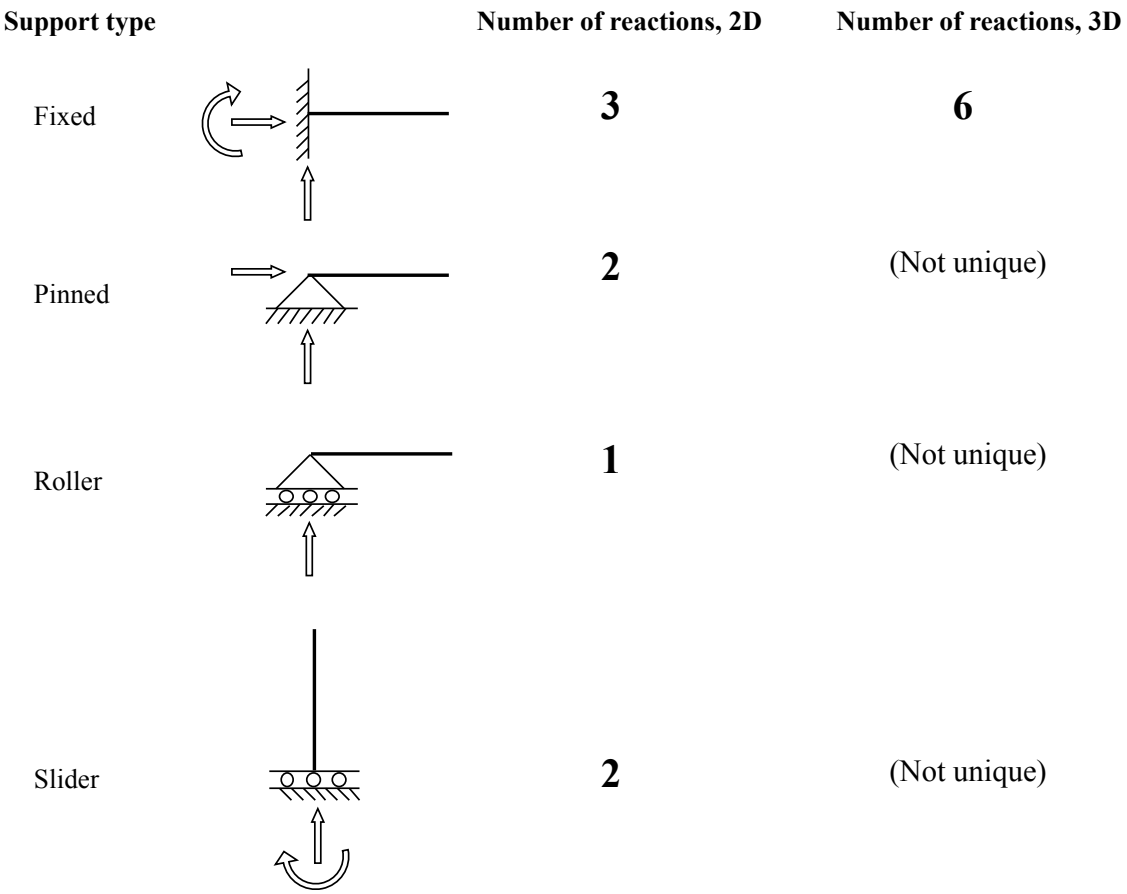


**Figure 1: Number of unknown forces and degrees of freedom for some joint types.**

The number of hinges, *h*, is obtained by counting the number of moment hinges and other types of section force releases in the structure. Essentially, a hinge releases one or more section forces from being transferred into one or more adjacent member. As a result, *h* counts the number of section forces that are prevented from transferring into adjacent

members. For example, an ordinary hinge, i.e., a moment hinge, in a continuous beam prevents the bending moment from being transferred through the hinge, thus $h$=1. Naturally, this reduces the number of unknowns because the bending moment is known to be zero at the hinge. It is emphasized that other releases are possible, where the shear or axial forces is prevented from transferring to an adjacent member.

In conclusion, Eq. (1) represents a powerful approach to determine the DSI and it works for all kinds of structures and configurations. Confusion in the application of the formula is sometimes related to the identification of members and joints in frame structures. For frames the number of members, $m$, and joints, $j$, is subjective in the sense that any member can be split into two by placing a joint somewhere on the member. However, this subjectivity does not affect the final DSI. Another potential source of confusion is mixed truss and frame structures. Here two approaches are possible. One is to consider all members as frame members when evaluating $f$. This adds shear forces and bending moments as unknowns even for the truss members, but this is counteracted by counting hinges, $h$, at the end of every truss member to ensure that these do not take any bending moment. The other approach, which is preferred by the author of this document, is to let $f$ be different for the different types of members, because they actually do have different numbers of section forces. In turn, the number of equations in each joint, $e$, is counted as a frame if there is any frame member coming into the joint.

## DSI by Counting DOFs

An important concept in advanced structural analysis is "degrees of freedom" (DOF), which is further explained later in this document. A method exists by which DOFs are counted to determine the DSI. This approach is closely related to the approach described above in the context of Eq. (1). This approach takes advantage of the fact that the number of DOFs in the structure equals the number of equilibrium equations in Eq. (1), namely $e{\cdot}j$, minus the number of support reactions, $s$. As a result, the equation

$$DSI = f \cdot m - DOF \tag{2}$$

counts the number of member forces minus the free degrees of freedom, which makes it equivalent to Eq. (1) with the following caveat; releases, such as moment hinges, are included by one of these two approaches:

- Count four DOFs at the joint where the hinge is located, i.e., two displacements and two rotations (adjust for 3D structures), and count unknown member forces as usual
- Count *two* DOFs at the joint where the hinge is located, i.e., the horizontal and vertical DOF (adjust for 3D structures), but *omit* the counting of bending moments in the adjacent members as unknown forces.

With that caveat the approaches in Eqs. (1) and (2) are essentially the same because the support reactions are included explicitly in the previous approach while they here are included implicitly as part of the DOF number.

## DSI by Counting Global Members

This approach is less versatile than the techniques described above, but it is popular in the fundamental textbooks because it better illustrates the meaning of DSI. The structure is divided into parts, none of which can include closed loops, which each are subjected to global equilibrium considerations. One number, usually denoted *r*, counts the total number of support reactions and internal forces at the "cuts" between parts of the structure. For example, a cut in a 2D frame member carries three unknown forces, while an ordinary hinge, which automatically constitutes a cut, carries two unknown forces. Another number, usually denoted *n*, counts the number of parts that the structure has been divided into. As a result, for 2D frame structures:

$$DSI = r - 3 \cdot n \tag{3}$$

because three equilibrium equations are available for each part, namely horizontal, vertical, and rotational equilibrium. This approach gives a good "feel" for the DSI because equilibrium is directly considered for each part, but it is more ad *hoc in* nature, and less suited for truss structures and complicated frame structures.

## External and Internal DSI

This fact is usually unimportant, but the DSI can actually include two types of indeterminacy. The external DSI is the number of unknown support reaction forces beyond the global equilibrium equations that can be established for the structure. For example, three equilibrium equations can be established for any 2D truss and frame structure; horizontal, vertical, and rotational. Thus, three support reaction forces can be determined; additional support reactions imply external indeterminacy. Similarly, for 3D structures there are six global equilibrium equations available; for these structures any support reactions beyond six implies external indeterminacy. The rest of the DSI number represents degrees internal indeterminacy.

## DSI vs. Stability

An unstable structure cannot be analysed and will immediately collapse. Instability implies that there are modes of deformation with zero stiffness, often referred to as "mechanisms." For 2D structural models it is often straightforward to see that a structure is unstable. Figure 2 shows examples of unstable structural models. The two hinges at the top of the frame combined with the pinned supports means that this structure will collapse sideways. The truss structure in Figure 2 is also unstable; a cross brace is required to make this a useful structure.

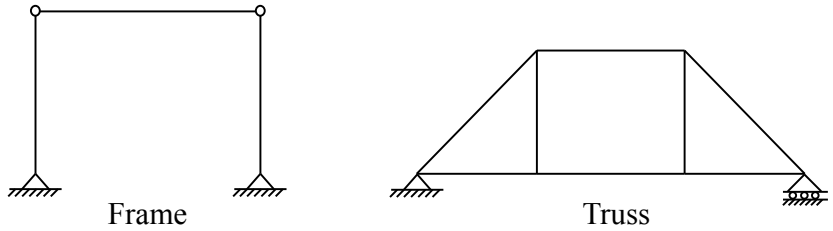


**Figure 2: Unstable frame and truss.**

In addition to visual inspection, there are a few indicators that may expose instabilities. One indicator is a negative value of the DSI. This is a sure indication that the structure is unstable, but unfortunately the DSI is not a reliable test of stability; a structure can be statically determinate and still unstable. Another approach, from advanced structural analysis, is to model the structure by the stiffness method and attempt to solve the resulting linear system of equilibrium equations. If the stiffness matrix is singular, i.e., it cannot be inverted, then instability may be the cause. It is understood from the imperfection of DSI as a stability-indicator and the advanced modelling involved in the stiffness method approach that visual inspection is the most important tool for the determination of a structure's stability

## DSI vs. DKI, Degrees of Freedom

While the DSI provides information about unknown member forces, the number of DOFs, sometimes referred to as the degree of kinematic indeterminacy (DKI), exposes the number of unknown joint displacements and rotations. In other words, DSI is a key number in force-based analysis methods for indeterminate structures, while DKI is the key figure in displacement-based methods. In fact, the DSI is the size of the flexibility matrix and DKI is the size of the stiffness matrix, for the flexibility methods and the stiffness method, respectively.

The DOFs are easier to determine than the DSI and even a computer can do it in a straightforward manner. This is why the stiffness method is implemented in all structural analysis software, while the flexibility method is not. Each joint, usually called node in the context of displacement-based methods, has a pre-defined number of DOFs. For example, a 2D structure has three DOFs per node: horizontal, vertical, and rotational motion. Similarly, a 3D structural model has six degrees of freedom per node: three displacements and three rotations. For truss structures the rotational DOFs are neglected altogether because they are associated with zero stiffness from the truss elements. Some structural analysis programs deal with trusses by first keeping all rotational degrees of freedom and later restraining them in the same way as nodes with boundary conditions are restrained.

When doing *hand calculations* there are two exceptions to the computer rule that every node has the same number of DOFs. The first exception is nodes at supports, where movement of the node is restrained. For example, in hand calculations a completely fixed node is counted to have zero DOFs, while the computer would count the standard number of DOFs and restrain them later. The second exception appears when axial deformations are neglected in the analysis of frame structures. This is quite common in hand calculations with the classical stiffness method because the axial deformation of frame members is usually insignificantly for practical purposes. Neglecting axial deformations requires careful consideration of the DOFs at each node, which is relatively simple for the trained human eye but difficult for a computer. Hence, in computer analysis it is easier to always account for axial deformations, while by hand one simply removes the DOFs that will experience zero displacement when the members do not deform axially.